

FINITE DIMENSIONAL REPRESENTATIONS OF W -ALGEBRAS

IVAN LOSEV

ABSTRACT. W -algebras of finite type are certain finitely generated associative algebras closely related to the universal enveloping algebras of semisimple Lie algebras. In this paper we prove a conjecture of Premet that gives an almost complete classification of finite dimensional irreducible modules for W -algebras. Also we study a relation between Harish-Chandra bimodules and bimodules over W -algebras.

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1. INTRODUCTION

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Address: Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA. E-mail: ivanlosev@math.mit.edu.

1.1. W-algebras. Let \mathfrak{g} be a finite dimensional semisimple Lie algebra over an algebraically closed field \mathbb{K} of characteristic zero and G be the simply connected algebraic group with Lie algebra \mathfrak{g} . Fix a nilpotent element $e \in \mathfrak{g}$ and let \mathbb{O} denote its adjoint orbit. Associated with the pair (\mathfrak{g}, e) is a certain associative unital algebra \mathcal{W} called the W -algebra (of finite type). In the special case when e is a principal nilpotent element this algebra appeared in Kostant's paper [K]. In this case the W -algebra is naturally isomorphic to the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U} := U(\mathfrak{g})$. In the general case, a definition of a W -algebra was given by Premet, [Pr1]. Since then W -algebras were extensively studied, see, for instance, [BGK], [BrK11]-[BrK13], [GG], [Gi2], [Lo1], [Pr2]-[Pr4].

Let us review Premet's definition briefly. The definition is recalled in more detail in Subsection 2.2.

To e one assigns a certain subalgebra $\mathfrak{m} \subset \mathfrak{g}$ consisting of nilpotent elements and of dimension $\frac{1}{2} \dim \mathbb{O}$, and also a character $\chi : \mathfrak{m} \rightarrow \mathbb{K}$. Set $\mathfrak{m}_\chi := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}\}$. The W -algebra \mathcal{W} associated with the pair (\mathfrak{g}, e) is, by definition, the quantum Hamiltonian reduction $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{ad m}} := \{a + \mathcal{U}\mathfrak{m}_\chi \mid [\mathfrak{m}, a] \subset \mathcal{U}\mathfrak{m}_\chi\}$. This algebra has the following nice features.

1) Choose an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} and set $Q := Z_G(e, h, f)$. There is an action of Q on \mathcal{W} by algebra automorphisms. Moreover, there is a Q -equivariant embedding $\mathfrak{q} := \text{Lie}(Q) \hookrightarrow \mathcal{W}$ such that the adjoint action of $\mathfrak{q} \subset \mathcal{W}$ on \mathcal{W} coincides with the differential of the action $Q : \mathcal{W}$.

2) There is a distinguished increasing exhaustive filtration (the Kazhdan filtration) $K_i \mathcal{W}, i \geq 0$, of \mathcal{W} with $K_0 \mathcal{W} = \mathbb{K}$. As Premet checked in [Pr1], the associated graded algebra is naturally identified with the algebra of regular functions on a transverse slice $S \subset \mathfrak{g}$ to \mathbb{O} called the *Slodowy slice*. The slice S can be defined as $e + \mathfrak{z}_{\mathfrak{g}}(f)$.

3) The space $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ has a natural structure of a \mathcal{U} - \mathcal{W} -bimodule. This allows to define the functor $N \mapsto \mathcal{S}(N)$ from the category of (left) \mathcal{W} -modules to the category of \mathcal{U} -modules: $\mathcal{S}(N) := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi) \otimes_{\mathcal{W}} N$. This functor defines an equivalence of $\mathcal{W}\text{-Mod}$ with the full subcategory of $\mathcal{U}\text{-Mod}$ consisting of all *Whittaker* \mathfrak{g} -modules, i.e., those, where the action of \mathfrak{m}_χ is locally nilpotent. The quasi-inverse functor is given by $M \mapsto M^{\mathfrak{m}_\chi} := \{m \in M \mid \xi m = \langle \chi, \xi \rangle m, \forall \xi \in \mathfrak{m}\}$. This was proved by Skryabin in the appendix to [Pr1].

1.2. Finite dimensional irreducible representations. One of the most important problems arising in representation theory of associative algebras is to classify their irreducible finite dimensional representations. Such representations are in one-to-one correspondence with *primitive* ideals of finite codimension; recall that a two-sided ideal is called primitive if it coincides with the annihilator of some irreducible module.

In [Pr2] Premet proposed to study the map $N \mapsto \text{Ann}_{\mathcal{U}} \mathcal{S}(N)$ from the set of all finite dimensional irreducible \mathcal{W} -modules to the set of primitive ideals in \mathcal{U} . He proved that the images consists of ideals, whose associated variety in \mathfrak{g} coincides with \mathbb{O} , and conjectured that any such primitive ideal can be represented in the form $\text{Ann}_{\mathcal{U}} \mathcal{S}(N)$. This conjecture was proved by Premet in [Pr3] under some mild restriction on an ideal, and by the author in [Lo1] in the full generality, alternative proofs were recently found by Ginzburg, [Gi2], and Premet, [Pr4]. Actually, the author obtained a more precise result. He constructed two maps $\mathcal{I} \mapsto \mathcal{I}^\dagger : \mathfrak{Id}(\mathcal{W}) \rightarrow \mathfrak{Id}(\mathcal{U}), \mathcal{J} \mapsto \mathcal{J}_\dagger : \mathfrak{Id}(\mathcal{U}) \rightarrow \mathfrak{Id}(\mathcal{W})$ between the sets $\mathfrak{Id}(\mathcal{W}), \mathfrak{Id}(\mathcal{U})$ of two-sided ideals of \mathcal{W}, \mathcal{U} . These two maps enjoy the following properties (see Theorem 3.1.1 for more details):

- (a) \mathcal{I}^\dagger is primitive whenever \mathcal{I} is. If, in addition, \mathcal{I} is of finite codimension, then the associated variety $V(\mathcal{U}/\mathcal{I}^\dagger)$ coincides with $\overline{\mathbb{O}}$.
- (b) $\text{Ann}_{\mathcal{W}}(N)^\dagger = \text{Ann}_{\mathcal{U}}(\mathcal{S}(N))$ for any \mathcal{W} -module N .
- (c) $\text{codim}_{\mathcal{W}} \mathcal{J}_\dagger = \text{mult}_{\overline{\mathbb{O}}}(\mathcal{U}/\mathcal{J})$ (see Subsection 1.6 for the definition of $\text{mult}_{\overline{\mathbb{O}}}$) provided $V(\mathcal{U}/\mathcal{J}) = \overline{\mathbb{O}}$.
- (d) If \mathcal{J} is primitive and $V(\mathcal{U}/\mathcal{J}) = \overline{\mathbb{O}}$, then $\{\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W}) | \mathcal{I}^\dagger = \mathcal{J}\}$ coincides with the set of all primitive ideals of \mathcal{W} containing \mathcal{J}_\dagger .

Here and below $\mathfrak{Id}_{fin}(\mathcal{W})$ denotes the set of all two-sided ideals of finite codimension in \mathcal{W} .

Premet suggested a stronger version of his existence conjecture including also a uniqueness statement (e-mail correspondence). The group Q acts naturally on $\mathfrak{Id}_{fin}(\mathcal{W})$. By 1) above, the unit component Q° of Q acts on $\mathfrak{Id}_{fin}(\mathcal{W})$ trivially, so the action of Q descends to that of the component group $C(e) := Q/Q^\circ$.

Conjecture 1.2.1 (Premet). For any primitive $\mathcal{J} \in \mathfrak{Id}_{\mathbb{O}}(\mathcal{U}) := \{\mathcal{J} \in \mathfrak{Id}(\mathcal{U}) | V(\mathcal{U}/\mathcal{J}) = \overline{\mathbb{O}}\}$ the set of all primitive ideals $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$ with $\mathcal{I}^\dagger = \mathcal{J}$ is a single $C(e)$ -orbit.

Note that irreducible \mathcal{W} -modules, whose annihilators are $C(e)$ -conjugate, are very much alike. In the representation theory of \mathcal{U} there are (complicated) techniques allowing to describe the set of primitive ideals in $\mathfrak{Id}_{\mathbb{O}}(\mathcal{U})$, see [Ja] for details. So Conjecture 1.2.1 provides an almost complete classification of irreducible finite dimensional representations of \mathcal{W} . This classification is complete whenever the action of $C(e)$ on $\mathfrak{Id}_{fin}(\mathcal{W})$ is trivial. This is the case, for example, when $\mathfrak{g} = \mathfrak{sl}_n$. Here $Q = Q^\circ Z(G)$ and $Z(G)$ acts trivially on \mathcal{W} . Here the classification was obtained by Brundan and Kleshchev, [BrKl2] by completely different methods (they used a relation between W -algebras and shifted Yangians).

In Subsection 4.2 we derive Conjecture 1.2.1 from the following statement.

Theorem 1.2.2 (Extended Premet's conjecture). *An element $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$ equals \mathcal{J}_\dagger for some $\mathcal{J} \in \mathfrak{Id}_{\mathbb{O}}(\mathcal{U})$ if and only if \mathcal{I} is $C(e)$ -invariant. If this is the case, then $\mathcal{I} = (\mathcal{I}^\dagger)_\dagger$.*

1.3. Harish-Chandra bimodules. In this paper we also obtain some results on a relation between Harish-Chandra \mathcal{U} -bimodules and \mathcal{W} -bimodules. The idea to study this relation was communicated to me by Ginzburg, his own approach is explained in [Gi2]. Recall that a \mathcal{U} -bimodule \mathcal{M} is said to be Harish-Chandra if it is finitely generated (as a bimodule) and the adjoint action of \mathfrak{g} on \mathcal{M} is locally finite.

On the W -algebra side we consider the category of Q -equivariant finite dimensional \mathcal{W} -bimodules. We say that a \mathcal{W} -bimodule \mathcal{N} is Q -equivariant, if it is equipped with a locally finite linear action of Q such that

- (1) The structure map $\mathcal{W} \otimes \mathcal{N} \otimes \mathcal{W} \rightarrow \mathcal{N}$ is Q -equivariant.
- (2) The differential of the Q -action (defined since the action is locally finite) coincides with the adjoint action of $\mathfrak{q} \subset \mathcal{W}$ on \mathcal{N} : $(\xi, n) \mapsto \xi n - n \xi$.

Q -equivariant finite dimensional \mathcal{W} -bimodules form a monoidal abelian category (tensor product is the tensor product of \mathcal{W} -bimodules), which we denote by $\text{HC}_{fin}^Q(\mathcal{W})$.

It turns out that the category $\text{HC}_{fin}^Q(\mathcal{W})$ is closely related to a certain subquotient of the category of Harish-Chandra \mathcal{U} -bimodules.

Namely, consider the abelian category $\text{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ of Harish-Chandra \mathcal{U} -bimodules \mathcal{M} whose associated variety $V(\mathcal{M})$ is contained in $\overline{\mathbb{O}}$. It has a Serre subcategory $\text{HC}_{\partial\mathbb{O}}(\mathcal{U})$ consisting of all Harish-Chandra bimodules \mathcal{M} with $V(\mathcal{M}) \subset \partial\mathbb{O} := \overline{\mathbb{O}} \setminus \mathbb{O}$. We can form the quotient category $\text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}) := \text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}) / \text{HC}_{\partial\mathbb{O}}(\mathcal{U})$.

The category $\mathrm{HC}(\mathcal{U})$ of all Harish-Chandra bimodules has a monoidal structure with respect to the tensor product of \mathcal{U} -bimodules. The subcategory $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ is closed with respect to tensor products (but does not contain a unit of $\mathrm{HC}(\mathcal{U})$). Clearly, the tensor product descends to $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$.

In [Gi2], Section 4, Ginzburg constructed an exact functor $\mathrm{HC}_{\mathbb{O}}(\mathcal{U}) \rightarrow \mathrm{HC}_{fin}^Q(\mathcal{W})$ (in fact, he did not consider Q -actions but his construction can be easily upgraded to the Q -equivariant setting, see Subsection 3.5). Roughly speaking, this functor should be close to an equivalence (but there are strong evidences that it is not, the actual situation should be much subtler, see the next subsection). In this paper we obtain some partial results towards this claim to be stated now.

In Subsection 3.4 we will construct functors $\mathcal{M} \rightarrow \mathcal{M}_{\dagger} : \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}) \rightarrow \mathrm{HC}_{fin}^Q(\mathcal{W}), \mathcal{N} \rightarrow \mathcal{N}^{\dagger} : \mathrm{HC}_{fin}^Q(\mathcal{W}) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$. The following theorem describes the properties of these two functors.

Theorem 1.3.1. (1) *The functor $\mathcal{M} \mapsto \mathcal{M}_{\dagger}$ is exact and left-adjoint to the functor $\mathcal{N} \mapsto \mathcal{N}^{\dagger}$. Moreover, $\mathcal{U}_{\dagger} = \mathcal{W}$ and for an ideal $\mathcal{J} \subset \mathcal{U}$ its image under the functor \bullet_{\dagger} coincides with the ideal \mathcal{J}_{\dagger} mentioned in the previous subsection.*
(2) *Let $\mathcal{M} \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$. Then $\dim \mathcal{M}_{\dagger} = \mathrm{mult}_{\overline{\mathbb{O}}}(\mathcal{M})$, and the kernel and the cokernel of the natural homomorphism $\mathcal{M} \rightarrow (\mathcal{M}_{\dagger})^{\dagger}$ lie in $\mathrm{HC}_{\partial\mathbb{O}}(\mathcal{U})$.*
(3) *$\mathcal{M} \rightarrow \mathcal{M}_{\dagger}$ is a tensor functor.*
(4) *$\mathrm{LAnn}(\mathcal{M})_{\dagger} = \mathrm{LAnn}(\mathcal{M}_{\dagger}), \mathrm{RAnn}(\mathcal{M})_{\dagger} = \mathrm{RAnn}(\mathcal{M}_{\dagger})$.*
(5) *The functor $\mathcal{M} \mapsto \mathcal{M}_{\dagger}$ gives rise to an equivalence of $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$ and some full subcategory in $\mathrm{HC}_{fin}^Q(\mathcal{W})$ closed under taking subquotients.*

Here $\mathrm{LAnn}, \mathrm{RAnn}$ denote the left and right annihilators of a bimodule.

We will see in Subsection 3.5 that our functor \bullet_{\dagger} coincides with that of Ginzburg.

Finally, let us state a corollary of Theorem 1.3.1 giving a sufficient condition for semisimplicity of an object in $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$. This corollary was suggested to the author by R. Bezrukavnikov. It will be proved in Subsection 4.2.

Corollary 1.3.2. *Let $\mathcal{M} \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ be such that $\mathrm{LAnn}(\mathcal{M}), \mathrm{RAnn}(\mathcal{M})$ are primitive ideals. Then the \mathcal{M} is semisimple in $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$.*

1.4. Further developments. An important open problem about the functor $\bullet_{\dagger} : \mathrm{HC}_{\mathbb{O}}(\mathcal{U}) \rightarrow \mathrm{HC}_{fin}^Q(\mathcal{W})$ is to describe its image.

Perhaps, the first question one can ask is to describe irreducible objects in $\mathrm{HC}_{fin}^Q(\mathcal{W})$ lying in the image of \bullet_{\dagger} . Further, a reasonable restriction is to consider only \mathcal{W} -bimodules with trivial left and right central characters. Here we use a natural identification of $\mathcal{Z}(\mathfrak{g})$ with the center of \mathcal{W} , see Subsection 2.2 for details. A nonzero finite dimensional \mathcal{W} -module with trivial central character exists if and only if \mathbb{O} is special in the sense of Lusztig.

In a subsequent paper we will prove the following result.

Theorem 1.4.1. *For any finite dimensional irreducible \mathcal{W} -modules N_1, N_2 with trivial central character there is a simple object $\mathcal{M} \in \mathrm{HC}_{\mathbb{O}}(\mathcal{U})$ such that $\mathrm{Hom}_{\mathbb{K}}(N_1, N_2)$ is a direct summand of \mathcal{M}_{\dagger} .*

We remark that Theorem 1.3.1 implies that \mathcal{M}_{\dagger} is simple in $\mathrm{HC}_{fin}^Q(\mathcal{W})$ and so is semisimple as a \mathcal{W} -bimodule. Theorem 1.4.1 can be generalized to any (possible singular) integral central character but it fails for a non-integral one.

The result mentioned in the previous paper establishes an interesting relation between the functor \bullet_{\dagger} and the work of Bezrukavnikov, Finkelberg and Ostrik, [BFO1],[BFO2]. In those papers they proved (among other things) that the following two monoidal categories are "almost isomorphic":

- The subcategory $\mathrm{HC}_{\mathbb{O}}^{ss}(\mathcal{U}_0)$ of all semisimple objects in $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$ with trivial left and right central characters; the claim that this category is closed with respect to the tensor product functor follows from Corollary 1.3.2.
- The category $\mathrm{Coh}^{A(\mathbb{O})}(Y \times Y)$, where $A(\mathbb{O})$ is the quotient of $C(e)$ defined by Lusztig, see [Lu], and Y is some finite set acted on by $A(\mathbb{O})$; the notation $\mathrm{Coh}^{A(\mathbb{O})}$ means the category of all $A(\mathbb{O})$ -equivariant sheaves of finite dimensional vector spaces; the tensor product on $\mathrm{Coh}^{A(\mathbb{O})}(Y \times Y)$ is given by convolution.

"Almost isomorphic" means that the monoidal categories in consideration become isomorphic after modifying the associativity isomorphisms for triple tensor products in one of them, but actually in almost all cases there is a genuine isomorphism.

In a subsequent paper we will prove that for Y we can take the set of (isomorphism classes of) irreducible finite dimensional \mathcal{W} -modules with trivial central character (and so, in particular, that the $C(e)$ -action on this set factors through $A(\mathbb{O})$). The first result in this direction is Theorem 1.4.1.

Another question one can pose is whether the techniques of the present paper are specific for the universal enveloping algebras of semisimple Lie algebras or they can be modified to study analogous questions for other algebras. At the moment, we know one important class of algebras, where this modification is possible: the symplectic reflection algebras (shortly, SRA) of Etingof and Ginzburg, [EG]. This is studied in our preprint [Lo2].

Namely, let V be a symplectic vector space and Γ be a finite group acting on V by linear symplectomorphisms. The symplectic reflection algebra \mathcal{H} is a certain flat deformation of the smash-product $SV \# \Gamma$, see [EG] for a precise definition. In the SRA setting the role of nilpotent orbits is played by symplectic leaves of V^*/Γ . The symplectic leaves are parametrized by conjugacy classes of stabilizers for the action of Γ on V . Namely, let \mathcal{L} be a symplectic leaf. Pick a point $b \in V$, whose image in V^*/Γ lies in \mathcal{L} . Then to \mathcal{L} we assign the stabilizer $\underline{\Gamma}$ of b in Γ , the point b and so its stabilizer are defined uniquely up to Γ -conjugacy. The role of Q is played by $N_{\Gamma}(\underline{\Gamma})$ and the role of $C(e)$ by $N_{\Gamma}(\underline{\Gamma})/\underline{\Gamma}$.

It turns out that for SRA we have complete analogs of Theorems 1.2.2, 1.3.1. Both construction of maps and functors for SRA and the proofs that these maps and functors have the required properties are very similar to (but more complicated technically than) those of the present paper.

1.5. Content of the paper. Techniques we use to prove Theorems 1.2.2, 1.3.1 are similar to those used in [Lo1] to construct maps $\mathcal{I} \mapsto \mathcal{I}^{\dagger}, \mathcal{J} \mapsto \mathcal{J}_{\dagger}$. We replace the W -algebra by a certain noncommutative algebra of formal power series and the universal enveloping with the algebra of polynomials inside the power series. Then Theorem 1.2.2 becomes a criterion for a two-sided ideal to be generated by polynomials. The functors in Theorem 1.3.1 are, roughly speaking, "completion" and "taking the \mathfrak{g} -finite part".

This paper is organized as follows. Section 2 contains some preliminary material. In its first subsection we review basic properties of deformation quantization, the key technique in the approach to W -algebras developed in [Lo1]. Also we recall some results on the existence of quantum moment maps. In Subsection 2.2 we recall two definitions of W -algebras: one due to Premet (in a variant of Gan-Ginzburg, [GG]) and one from [Lo1]. Subsection 2.3

recalls (and, in fact, proves a stronger version of) a basic result we used in [Lo1] to study W-algebras, the so called decomposition theorem. In Subsection 2.4 we prove some technical results on completions of quantum algebras. Finally in Subsection 2.5 we discuss the notion of Harish-Chandra bimodules for quantum algebras in interest.

Section 3 is devoted mostly to constructing the functors between the categories of bimodules. In Subsection 3.1 we recall the definitions of the maps $\mathcal{I} \mapsto \mathcal{I}^\dagger, \mathcal{J} \mapsto \mathcal{J}_\dagger$. Subsection 3.2 is technical, there we prove some results on homogeneous vector bundles to be used both in the construction of functors and in the proofs of the main theorems (roughly, as induction steps). Subsections 3.3, 3.4 form a central part of the section: there we define the functors between the categories of bimodules and study basic properties of the functors. At first, we do this on the level of quantum algebras, Subsection 3.3, and then on the level of \mathcal{U}, \mathcal{W} , verifying that our constructions essentially do not depend on filtrations. In Subsection 3.5 we compare our construction with Ginzburg's, [Gi2].

In Section 4 we complete the proofs of the two main theorems. In the first subsection we state some auxiliary result (Theorem 4.1.1), which is a straightforward generalization of Theorem 1.2.2 and also the most difficult part of Theorem 1.3.1. Then in the second subsection we complete the proofs of Theorems 1.2.2, 1.3.1 using Theorem 4.1.1.

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1.6. Notation and conventions. Let us explain several notions used below in the text.

Adjoint actions on bimodules. Let \mathcal{A} be an associative algebra and \mathcal{M} be an \mathcal{A} -bimodule. For $a \in \mathcal{A}$ we write $[a, m] := am - ma$.

Associated varieties and multiplicities. Let \mathcal{A} be an associative algebra equipped with an increasing filtration $F_i \mathcal{A}$. We suppose that $\text{gr } \mathcal{A} := \sum_{i \in \mathbb{Z}} F_i \mathcal{A} / F_{i-1} \mathcal{A}$ is a Noetherian commutative algebra. Now let \mathcal{M} be a filtered \mathcal{A} -module such that $\text{gr } \mathcal{M}$ is a finitely generated $\text{gr } \mathcal{A}$ -module. By the *associated variety* $V(\mathcal{M})$ of \mathcal{M} we mean the support of $\text{gr } \mathcal{M}$ in $\text{Spec}(\mathcal{A})$. Moreover, since $\text{gr } \mathcal{M}$ is finitely generated, there is a $\text{gr } \mathcal{A}$ -module filtration $\text{gr } \mathcal{M} = M_0 \supset M_1 \supset M_2 \supset \dots \supset M_k = \{0\}$ such that $M_i / M_{i+1} = (\text{gr } \mathcal{A}) / \mathfrak{p}_i$, where \mathfrak{p}_i is a prime ideal in $\text{gr } \mathcal{A}$. For an irreducible component Y of $V(\mathcal{M})$ we write $\text{mult}_Y \mathcal{M}$ for the number of indexes i with $M_i / M_{i+1} \cong \mathcal{A} / \mathfrak{p}_Y$, where \mathfrak{p}_Y is the prime ideal corresponding to Y . The number $\text{mult}_Y \mathcal{M}$ is called the *multiplicity* of \mathcal{M} at Y . It is known that $V(\mathcal{M})$ and $\text{mult}_Y \mathcal{M}$ do not depend on the choices we made. When \mathcal{M} is an \mathcal{A} -bimodule, $V(\mathcal{M})$ stands for the associated variety of \mathcal{M} regarded as a left \mathcal{A} -module.

Now let \mathcal{A}_\hbar be an associative $\mathbb{K}[\hbar]$ -algebra such that $\mathcal{A}_\hbar / (\hbar)$ is Noetherian and commutative. For a finitely generated \mathcal{A}_\hbar -module \mathcal{M}_\hbar let $V(\mathcal{M}_\hbar)$ stand for the support of $\mathcal{M}_\hbar / \hbar \mathcal{M}_\hbar$ in $\text{Spec}(\mathcal{A}_\hbar / (\hbar))$. For a component $Y \subset V(\mathcal{M}_\hbar)$ one sets $\text{mult}_Y \mathcal{M}_\hbar := \text{mult}_Y(\mathcal{M}_\hbar / \hbar \mathcal{M}_\hbar)$.

Locally finite parts. Let \mathfrak{g} be some Lie algebra and let M be a module over \mathfrak{g} . By the locally finite (shortly, l.f.) part of M we mean the sum of all finite dimensional \mathfrak{g} -submodules of M . Similarly, if G is an algebraic group acting on M , then the G -locally finite part of M is the sum of all finite dimensional G -submodules, where the action of G is algebraic.

\hbar -saturated subspaces. Let V be a $\mathbb{K}[\hbar]$ -module. We say that a submodule $U \subset V$ is \hbar -saturated if $\hbar v \in U$ implies $v \in U$ for all $v \in V$.

Group actions on algebras, schemes, modules etc. All algebraic group actions considered in this paper are either algebraic or pro-algebraic. In the case of algebras or modules "algebraic" means "locally finite" (see above). "Pro-algebraic" means that an

algebra is an inverse limit of locally finite G -modules. In particular, given an action of an algebraic group G on a scheme, for any $\xi \in \mathfrak{g}$ the velocity vector field ξ_* on X is defined.

Let A be an algebra equipped with an action of a group G by algebra automorphisms. By a G -equivariant A -module we mean an A -module M equipped with a G -action such that the structure map $A \otimes M \rightarrow M$ is G -equivariant.

The notion of a G -weakly equivariant A -bimodule (compare with the terminology used for D -modules) we mean a G -equivariant $A \otimes A^{op}$ -module. The notion of a G -equivariant bimodule is different, compare with Subsection 1.3.

Below we gather some notation used in the paper.

$\widehat{\otimes}$	the completed tensor product of complete topological vector spaces/modules.
(a_1, \dots, a_k)	the two-sided ideal in an associative algebra generated by elements a_1, \dots, a_n .
A_χ^\wedge	the completion of a commutative algebra A with respect to the maximal ideal of a point $\chi \in \text{Spec}(A)$.
$\text{Ann}_{\mathcal{A}}(\mathcal{M})$	the annihilator of an \mathcal{A} -module \mathcal{M} in an algebra \mathcal{A} .
$\text{Der}(A)$	the Lie algebra of derivations of an algebra A .
$G *_H V$	$:= (G \times V)/H$: the homogeneous vector bundle over G/H with fiber V .
$g *_H v$	the class of $(g, v) \in G \times V$ in $G *_H V$.
G_x	the stabilizer of x in G .
$\text{Grk}(\mathcal{A})$	the Goldie rank of a prime Noetherian algebra \mathcal{A} .
$\text{gr } \mathcal{A}$	the associated graded vector space of a filtered vector space \mathcal{A} .
$I(Y)$	the ideal in $\mathbb{K}[X]$ consisting of all functions vanishing on Y for a subvariety Y in an affine variety X .
$\mathfrak{Id}(\mathcal{A})$	the set of all (two-sided) ideals of an algebra \mathcal{A} .
$\mathcal{M}_{\mathfrak{g}-l.f.}$	the locally finite part of a \mathfrak{g} -module \mathcal{M} .
$R_h(\mathcal{A})$	$:= \bigoplus_{i \in \mathbb{Z}} \hbar^i F_i \mathcal{A}$: the Rees vector space of a filtered vector space \mathcal{A} .
$U(\mathfrak{g})$	the universal enveloping algebra of a Lie algebra \mathfrak{g} .
$V(\mathcal{M})$	the associated variety of \mathcal{M} .
$\mathcal{Z}(\mathfrak{g})$	the center of $U(\mathfrak{g})$.
$\Gamma(X, \mathcal{F})$	the space of global sections of a sheaf \mathcal{F} on X .

2. PRELIMINARIES

2.1. Deformation quantization and quantum comoment maps. Let A be a commutative associative \mathbb{K} -algebra with unit equipped with a Poisson bracket.

Definition 2.1.1. A map $*$: $A \otimes_{\mathbb{K}} A \rightarrow A[[\hbar]]$, $f * g = \sum_{i=0}^{\infty} D_i(f, g) \hbar^{2i}$ is called a *star-product* if it satisfies the following conditions:

- (*1) A natural $(\mathbb{K}[[\hbar]]$ -bilinear) extension of $*$ to $A[[\hbar]] \otimes_{\mathbb{K}[[\hbar]]} A[[\hbar]]$ is associative, i.e., $(f * g) * h = f * (g * h)$ for all $f, g, h \in A$, and $1 \in A$ is a unit for $*$.
- (*2) $f * g - fg \in \hbar^2 A[[\hbar]]$, $f * g - g * f - \hbar^2 \{f, g\} \in \hbar^4 A[[\hbar]]$ for all $f, g \in A$ or, equivalently, $D_0(f, g) = fg$, $D_1(f, g) - D_1(g, f) = \{f, g\}$.

Star-products we deal with in this paper will satisfy the following additional property.

- (*3) D_i is a bidifferential operator of order at most i in each variable ("bidifferential of order at most i " means that for any $f \in A$ both maps $g \mapsto D_i(g, f)$ and $g \mapsto D_i(f, g)$ are differential operators of order at most i).

Note that usually a star-product is written as $f * g = \sum_{i=0}^{\infty} D_i(f, g) \hbar^i$ with $f * g - g * f - \hbar\{f, g\} \in \hbar^2 A[[\hbar]]$. The reason why we use \hbar^2 instead of \hbar is that our choice is better compatible with the Rees algebra construction, which is used to pass from a filtered \mathbb{K} -algebra to a graded $\mathbb{K}[[\hbar]]$ -algebra.

When we consider $A[[\hbar]]$ as an algebra with respect to the star-product, we call it a *quantum algebra*. If $A[\hbar]$ is a subalgebra in $A[[\hbar]]$ with respect to $*$, then we say that $*$ is a *polynomial star-product*, $A[\hbar]$ is also called a quantum algebra.

Example 2.1.2 (The Weyl algebra \mathbf{A}_\hbar). Let $X = V$ be a finite-dimensional vector space equipped with a constant nondegenerate Poisson bivector P . The *Moyal-Weyl* star-product on $A := \mathbb{K}[V]$ is defined by

$$f * g = \exp\left(\frac{\hbar^2}{2} P\right) f(x) \otimes g(y)|_{x=y}.$$

Here P is considered as an element of $V \otimes V$. This space acts naturally on $\mathbb{K}[V] \otimes \mathbb{K}[V]$ (by contractions). The quantum algebra $\mathbf{A}_\hbar := \mathbb{K}[V][\hbar]$ is called the (*homogeneous*) *Weyl algebra*.

Remark 2.1.3. Actually, for a Fedosov star-product $*$ one has $D_i(f, g) = (-1)^i D_i(g, f)$. This is proved, for instance, in [BW], Lemma 3.3. In particular, $D_1(f, g) = \frac{1}{2}\{f, g\}$.

Now we discuss group actions on quantum algebras.

Let G be an algebraic group acting on A by automorphisms. It makes sense to speak about G -invariant star-products (\hbar is supposed to be G -invariant).

Recall that a G -equivariant linear map $\xi \mapsto H_\xi : \mathfrak{g} := \text{Lie}(G) \rightarrow A$ is said to be a *comoment map* if $\{H_\xi, \bullet\} = \xi_*$ for any $\xi \in \mathfrak{g}$. The action of G on A equipped with a comoment map is called *Hamiltonian*. In the case when A is finitely generated define the *moment map* $\mu : \text{Spec}(A) \rightarrow \mathfrak{g}^*$ to be the dual map to the comoment map $\mathfrak{g} \mapsto A$.

In the quantum situation there is an analog of a comoment map defined as follows: a G -equivariant linear map $\mathfrak{g} \rightarrow A[[\hbar]]$ is said to be a *quantum comoment map* if $[\hat{H}_\xi, \bullet] = \hbar^2 \xi_*$ for all $\xi \in \mathfrak{g}$.

Now let \mathbb{K}^\times act on A , $(t, a) \mapsto t.a$, by automorphisms. For instance, if A is graded, $A := \bigoplus_{i \in \mathbb{Z}} A_i$, we can consider the action coming from the grading: $t.a = t^i a, a \in A_i$. Consider the action $\mathbb{K}^\times : A[[\hbar]]$ given by $t. \sum_{j=0}^{\infty} a_j \hbar^j = \sum_{j=0}^{\infty} t^j (t.a_j) \hbar^j$. If \mathbb{K}^\times acts by automorphisms of $*$, then we say that $*$ is *homogeneous*. Clearly, $*$ is homogeneous if and only if the map $D_l : A \otimes A \rightarrow A$ is homogeneous of degree $-2l$.

The following theorem on existence of star-products and quantum comoment maps incorporates results of Fedosov, [F1]-[F3], in the form we need.

Theorem 2.1.4. *Let X be a smooth affine variety equipped with*

- *a symplectic form ω ,*
- *a Hamiltonian action of a reductive group G , $\xi \mapsto H_\xi$ being a comoment map,*
- *and an action of the one-dimensional torus \mathbb{K}^\times by G -equivariant automorphisms such that $t.\omega = t^2\omega, t.H_\xi = t^2 H_\xi$.*

Then there exists a G -invariant homogeneous star-product $$ on $\mathbb{K}[X]$ satisfying the additional condition (*3) and such that $\xi \mapsto H_\xi$ is a quantum comoment map.*

For instance, in Example 2.1.2, $*$ satisfies the conditions of Theorem 2.1.4 with $G = \mathrm{Sp}(V)$ and the action of \mathbb{K}^\times given by $t.v = t^{-1}v$. Note that $H_\xi(v) = \frac{1}{2}\omega(\xi v, v)$, $\xi \in \mathfrak{sp}(V)$, $v \in V$.

Fedosov proved an analog of Theorem 2.1.4 in the C^∞ -setting. The explanation why his results (except that on a quantum comoment map) hold in the algebraic setting together with references can be found in [Lo1], Subsection 2.2. The statement on the quantum comoment map is Theorem 2 in [F3]. Its proof can be transferred to the algebraic setting directly.

Note also that, since $*$ is G -invariant, we get a well-defined star-product on $\mathbb{K}[X]^G$. In this way, taking $X = T^*G$ and replacing G with $G \times G$, one gets a G -invariant star-product on $S(\mathfrak{g}) = \mathbb{K}[\mathfrak{g}^*]$. The corresponding quantum algebra will be denoted by \mathcal{U}_\hbar . This notation is justified by the observation that $\mathcal{U}_\hbar/(\hbar - 1) \cong \mathcal{U}$, see [Lo1], Example 2.2.4, for details. We will encounter another example of this construction in the following subsection.

2.2. W -algebras. In this subsection we review the definitions of W -algebras due to Premet, [Pr1], and the author, [Lo1].

Recall that a nilpotent element $e \in \mathfrak{g}$ is fixed and G denotes the simply connected algebraic group with Lie algebra \mathfrak{g} , $\mathbb{O} := Ge$. Choose an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} and set $Q := Z_G(e, h, f)$. Also introduce a grading on \mathfrak{g} by eigenvalues of $\mathrm{ad} h$: $\mathfrak{g} := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, $\mathfrak{g}(i) := \{\xi \in \mathfrak{g} \mid [h, \xi] = i\xi\}$. Since h is the image of a coroot under a Lie algebra homomorphism $\mathfrak{sl}_2 \rightarrow \mathfrak{g}$, we see that there is a unique one-parameter subgroup $\gamma : \mathbb{K}^\times \rightarrow G$ with $d_1\gamma(1) = h$.

The Killing form (\cdot, \cdot) on \mathfrak{g} allows to identify $\mathfrak{g} \cong \mathfrak{g}^*$, let $\chi = (e, \bullet)$ be an element of \mathfrak{g}^* corresponding to e . Identify \mathbb{O} with $G\chi$. Note that χ defines a symplectic form ω_χ on $\mathfrak{g}(-1)$ as follows: $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$. Fix a lagrangian subspace $l \subset \mathfrak{g}(-1)$ with respect to ω_χ and set $\mathfrak{m} := l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$. Define the affine subspace $\mathfrak{m}_\chi \subset \mathfrak{g}$ as in Subsection 1.1. Then, by definition, the W -algebra \mathcal{W} associated with (\mathfrak{g}, e) is $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\mathrm{ad} \mathfrak{m}} := \{a + \mathcal{U}\mathfrak{m}_\chi \mid [a, \mathfrak{m}] \subset \mathcal{U}\mathfrak{m}_\chi\}$.

Let us introduce a filtration on \mathcal{W} . We have the standard PBW filtration on \mathcal{U} (by the order of a monomial) denoted by $F_i^{st} \mathcal{U}$. The *Kazhdan filtration* $K_i \mathcal{U}$ is defined by $K_i \mathcal{U} := \sum_{2k+j \leq i} F_k^{st} \mathcal{U} \cap \mathcal{U}(j)$, where $\mathcal{U}(j)$ is the eigenspace of $\mathrm{ad} h$ on \mathcal{U} with eigenvalue j . Note that the associated graded algebra of \mathcal{U} with respect to the Kazhdan filtration is still naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$. Being a subquotient of \mathcal{U} , the W -algebra \mathcal{W} inherits the Kazhdan filtration (denoted by $K_i \mathcal{W}$). It is easy to see that $K_0 \mathcal{U} \subset \mathcal{U}\mathfrak{m}_\chi + \mathbb{K}$ and hence $K_0 \mathcal{W} = \mathbb{K}$.

There are two disadvantages of this definition. First, formally it depends on a choice of $l \subset \mathfrak{g}(-1)$. Second, one cannot see an action of Q on \mathcal{W} from it. Both disadvantages are remedied by a ramification of Premet's definition given by Gan and Ginzburg in [GG]. Namely, they checked that there is a natural isomorphism $(\mathcal{U}/\mathcal{U}\mathfrak{g}_{\leq -2, \chi})^{\mathrm{ad} \mathfrak{g}_{\leq -1}} \rightarrow \mathcal{W}$, where $\mathfrak{g}_{\leq k} := \sum_{i \leq k} \mathfrak{g}(i)$, $\mathfrak{g}_{\leq -2, \chi} := \{\xi - \langle \chi, \xi \rangle \mid \xi \in \mathfrak{g}_{\leq -2}\}$. Since all $\mathfrak{g}_{\leq k}$ are Q -stable, the group Q acts naturally on $(\mathcal{U}/\mathcal{U}\mathfrak{g}_{\leq -2, \chi})^{\mathrm{ad} \mathfrak{g}_{\leq -1}}$ and it is clear that the action is by algebra automorphisms. Also Premet checked in [Pr2] that there is an inclusion $\mathfrak{q} \hookrightarrow \mathcal{W}$ compatible with the action of Q in the sense explained in the Introduction.

Finally, note that there is a natural homomorphism $\mathcal{Z}(\mathfrak{g}) \hookrightarrow \mathcal{U}^{\mathrm{ad} \mathfrak{m}} \rightarrow (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\mathrm{ad} \mathfrak{m}}$. Premet checked in [Pr2] that it is injective and identifies $\mathcal{Z}(\mathfrak{g})$ with the center of \mathcal{W} .

Now let us recall the definition of \mathcal{W} given in [Lo1]. Define the Slodowy slice $S := e + \mathfrak{z}_{\mathfrak{g}}(f)$. It will be convenient for us to consider S as a subvariety in \mathfrak{g}^* via the identification $\mathfrak{g} \cong \mathfrak{g}^*$. Define the *Kazhdan* action of \mathbb{K}^\times on \mathfrak{g}^* by $t.\alpha = t^{-2}\gamma(t)\alpha$ for $\alpha \in \mathfrak{g}^*(i) := \mathfrak{g}(i)^*$. This action preserves S and, moreover, $\lim_{t \rightarrow \infty} t.s = \chi$ for all $s \in S$. Define the *equivariant Slodowy slice* $X := G \times S$. The variety X is naturally embedded into $T^*G = G \times \mathfrak{g}^*$.

Here we use the trivialization of T^*G by left-invariant 1-forms to identify T^*G with $G \times \mathfrak{g}^*$. Equip T^*G with a \mathbb{K}^\times -action given by $t.(g, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha)$ and with a Q -action by $q.(g, \alpha) = (gq^{-1}, q\alpha)$, $q \in Q, g \in G, \alpha \in \mathfrak{g}^*$. The equivariant Slodowy slice is stable under both actions. The action of $G \times Q$ on T^*G (and hence on X) is Hamiltonian with a moment map μ given by $\langle \mu(g, \alpha), (\xi, \eta) \rangle = \langle \text{Ad}(g)\alpha, \xi \rangle + \langle \alpha, \eta \rangle$, $\xi \in \mathfrak{g}, \eta \in \mathfrak{q}$.

According to [Lo1], Subsection 3.1, the Fedosov star-product on $\mathbb{K}[X]$ is polynomial. So we have a quantum algebra $\widetilde{\mathcal{W}}_h := \mathbb{K}[X][\hbar]$ (the equivariant W-algebra).

Taking G -invariants in $\widetilde{\mathcal{W}}_h$, we get a homogeneous Q -equivariant star-product $*$ on $\mathcal{W}_h := \mathbb{K}[S][\hbar]$ together with a quantum comoment map $\mathfrak{q} \rightarrow \mathbb{K}[S]$. This map is injective because the action of Q on X is free. Note that the grading on \mathcal{W}_h induces a filtration on $\mathcal{W}_h/(\hbar-1)$. Also note that the quantum comoment map $\mathfrak{g} \rightarrow \widetilde{\mathcal{W}}_h$ gives rise to a homomorphism $\mathcal{U}_h^G \rightarrow \mathcal{W}_h$. So we get a homomorphism $\mathcal{Z}(\mathfrak{g}) \hookrightarrow \mathcal{W}_h/(\hbar-1)$. It is not difficult to check that this homomorphism is an isomorphism of $\mathcal{Z}(\mathfrak{g})$ onto the center of \mathcal{W} but we will not give a direct proof of this.

Theorem 2.2.1. *There is a Q -equivariant isomorphism $\mathcal{W}_h/(\hbar-1) \cong \mathcal{W}$ of filtered algebras intertwining the homomorphisms from $\mathcal{Z}(\mathfrak{g})$.*

Almost for sure, one can assume, in addition, that an isomorphism intertwines also the embeddings of \mathfrak{q} . However, a slightly weaker claim follows from the Q -equivariance: namely, that the embeddings of \mathfrak{q} into $\mathcal{W} \cong \mathcal{W}_h/(\hbar-1)$ differ by a character of \mathfrak{q} .

A version of this theorem, which did not take the embeddings of $\mathcal{Z}(\mathfrak{g})$ to account, was proved in [Lo1], Corollary 3.3.3. To see that this isomorphism intertwines the maps from $\mathcal{Z}(\mathfrak{g})$ one can argue as follows. According to [Lo1], Remark 3.1.3, we have a G -equivariant isomorphism of $\widetilde{\mathcal{W}}_h/(\hbar-1)$ and the algebra $\mathcal{D}(G, e) := (\mathcal{D}(G)/\mathcal{D}(G)\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$, where \mathfrak{m} is embedded into $\mathcal{D}(G)$ via the velocity vector field map for the right G -action. For both algebras we have quantum comoment maps $\mathfrak{g} \rightarrow \widetilde{\mathcal{W}}_h/(\hbar-1), \mathcal{D}(G, e)$. This G -equivariant isomorphism intertwines the quantum comoment maps because G is semisimple. The isomorphism in Theorem 2.2.1 is obtained by restricting this G -equivariant isomorphism to the G -invariants. So the maps from $\mathcal{Z}(\mathfrak{g})$ are indeed intertwined.

2.3. Decomposition theorem. In a sense, the decomposition theorem is a basic result about W-algebras. In a sentence, it says that, up to completions, the universal enveloping algebra is decomposed into the tensor product of the W-algebra and of a Weyl algebra. We start with an equivariant version of this theorem.

Apply Theorem 2.1.4 to X, T^*G . We get a $G \times Q$ -equivariant homogeneous star-product $*$ on X and T^*G together with quantum comoment maps $\mathfrak{g} \times \mathfrak{q} \rightarrow \mathbb{K}[X][[\hbar]], \mathbb{K}[T^*G][[\hbar]]$.

Since the star-products on both $\mathbb{K}[X][[\hbar]]$ and $\mathbb{K}[T^*G][[\hbar]]$ are differential, we can extend them to the completions $\mathbb{K}[X]_{Gx}^\wedge[[\hbar]], \mathbb{K}[T^*G]_{Gx}^\wedge[[\hbar]]$ along the G -orbit Gx , where $x = (1, \chi)$. These algebras come equipped with natural topologies – the topologies of completions.

Set $V := [\mathfrak{g}, f]$. Equip V with the symplectic form $\omega(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$, the action of $\mathbb{K}^\times : t.v = \gamma(t)^{-1}v$ and the natural action of Q .

Theorem 2.3.1. *There is a $G \times Q \times \mathbb{K}^\times$ -equivariant $\mathbb{K}[[\hbar]]$ -linear isomorphism*

$$\Phi_h : \mathbb{K}[T^*G]_{Gx}^\wedge[[\hbar]] \rightarrow \mathbb{K}[X]_{Gx}^\wedge[[\hbar]] \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathbb{K}[V^*]_0^\wedge[[\hbar]]$$

intertwining the quantum comoment maps from $\mathfrak{g} \times \mathfrak{q}$.

Here $\widehat{\otimes}$ stands for the completed tensor product (i.e., we consider the tensor product of the topological algebras $\mathbb{K}[X]_{Gx}^\wedge[[\hbar]]$, $\mathbb{K}[V^*]_0^\wedge[[\hbar]]$ and then complete this tensor product with respect to the induced topology).

Proof. Recall from the proof of Theorem 3.3.1 in [Lo1] that there is a $G \times Q \times \mathbb{K}^\times$ -equivariant isomorphism $\varphi : \mathbb{K}[T^*G]_{Gx}^\wedge \rightarrow \mathbb{K}[X]_{Gx}^\wedge \widehat{\otimes} \mathbb{K}[V^*]_0^\wedge$ of Poisson algebras. This automorphism is $G \times Q$ -equivariant so it intertwines the classical comoment maps perhaps up to a character of $G \times Q$. Let us remark that the functions H_ξ have degree 2 with respect to the \mathbb{K}^\times -action. Since φ is \mathbb{K}^\times -equivariant, and the centers of our algebras consist of scalars, we see that the character has to vanish. Identify $A := \mathbb{K}[T^*G]_{Gx}^\wedge$ and $\mathbb{K}[X]_{Gx}^\wedge \widehat{\otimes} \mathbb{K}[V^*]_0^\wedge$ by means of φ . Then $\Phi_\hbar = id + \sum_{i=1}^\infty T_i \hbar^{2i}$, where T_i is a $G \times Q$ -invariant differential operator of degree $-2i$ with respect to \mathbb{K}^\times . We need to check that $\Phi_\hbar(H_\xi) = H_\xi$.

We claim that T_1 is a derivation of A . Indeed, let $*$, $'$ denote the two star-products on our algebra. Then consider the equality $\Phi_\hbar(f) *' \Phi_\hbar(g) = \Phi_\hbar(f * g)$, $f, g \in A$, modulo \hbar^4 . We get $(f + T_1(f)\hbar^2)(g + T_1(g)\hbar^2) + \frac{1}{2}\{f, g\}\hbar^2 \equiv fg + \frac{1}{2}\{f, g\} + T_1(fg)\hbar^2$ whence the claim. Now considering the skew-symmetric parts of the coefficients of \hbar^4 and using Remark 2.1.3, we see that T_1 annihilates the Poisson bracket.

Let η be the vector field on $(T^*G)_{Gx}^\wedge$ corresponding to T_1 . Since $H_{DR}^1(T^*G_{Gx}^\wedge) = H_{DR}^1(G) = \{0\}$, we see that η is a Hamiltonian vector field. So $T_1 = \{f, \cdot\}$ for some $G \times Q$ -equivariant function f . So $T_1(H_\xi) = \{f, H_\xi\} = 0$ for any $\xi \in \mathfrak{g} \times \mathfrak{q}$. In other words, $\Phi_\hbar(H_\xi) - H_\xi \in \hbar^4 A$. On the other hand, $\Phi_\hbar(H_\xi) - H_\xi$ is a central element (hence lies in $\mathbb{K}[[\hbar]]$) and has degree 2 with respect to the torus action. So this element is zero. \square

Remark 2.3.2. In the proof of the theorem we used the semisimplicity of G . However, Theorem 2.3.1 holds for a general reductive group too (we will need this situation in a subsequent paper). To show this suppose, at first, that $G = Z \times G_0$, where $Z := Z(G)^\circ$, $G_0 := (G, G)$. Then $T^*G = T^*Z \times T^*G_0$ and $X = T^*Z \times X_0$, where X_0 is the equivariant Slodowy slice for G_0 . Also we have the decomposition $Q = Z \times Q_0$, $Q_0 := Q \cap G_0$. Choose a $Z \times \mathbb{K}^\times$ -invariant symplectic connection on T^*Z , and $G_0 \times Q_0 \times \mathbb{K}^\times$ -invariant symplectic connections on $T^*(G, G)$ and X_0 . Equip T^*G and X with the products of the corresponding connections. Then for Φ_\hbar we can take the product of the identity isomorphism $\mathbb{K}[T^*Z]_Z^\wedge[[\hbar]] \rightarrow \mathbb{K}[T^*Z]_Z^\wedge[[\hbar]]$ and an isomorphism

$$\mathbb{K}[T^*G_0]_{G_0x}^\wedge[[\hbar]] \rightarrow \mathbb{K}[X_0]_{G_0x}^\wedge[[\hbar]] \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathbb{K}[V^*]_0^\wedge[[\hbar]],$$

satisfying the conditions of the theorem.

In general, to get the decomposition $G = Z \times G_0$ one needs to replace G with a covering. So let $G = (Z \times G_0)/\Gamma$, where Γ is a finite central subgroup in $Z \times G_0$. However, an isomorphism Φ_\hbar from the previous paragraph is Γ -equivariant. Its restriction to Γ -invariants has the required properties.

Consider the quantum algebras $\mathcal{U}_\hbar := \mathbb{K}[\mathfrak{g}^*][\hbar] = \mathbb{K}[T^*G][\hbar]^G$, $\mathcal{U}_\hbar^\wedge := \mathbb{K}[\mathfrak{g}^*]_\chi^\wedge[[\hbar]]$, $\mathbf{A}_\hbar^\wedge := \mathbb{K}[V^*]_0^\wedge[[\hbar]]$, $\mathcal{W}_\hbar := \mathbb{K}[S][\hbar]$, $\mathcal{W}_\hbar^\wedge := \mathbb{K}[S]_\chi^\wedge[[\hbar]]$ and finally $\mathbf{A}_\hbar^\wedge(\mathcal{W}_\hbar^\wedge) := \mathbf{A}_\hbar^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{W}_\hbar^\wedge$.

Restricting Φ_\hbar from Theorem 2.3.1 to the G -invariants, we get a Q -and \mathbb{K}^\times -equivariant isomorphism $\Phi_\hbar : \mathcal{U}_\hbar^\wedge \rightarrow \mathbf{A}_\hbar^\wedge(\mathcal{W}_\hbar^\wedge)$ of topological $\mathbb{K}[[\hbar]]$ -algebras. Till the end of the paper we fix this isomorphism.

2.4. Completions of quantum algebras. Let A be a finitely generated Poisson algebra, and $\mathcal{A}_\hbar = A[[\hbar]]$ be a quantum algebra with a star-product $f * g = \sum_{i=0}^\infty D_i(f, g)\hbar^{2i}$ satisfying

the condition (*3). Suppose that A is graded, $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with $\{A_i, A_j\} \subset A_{i+j-2}$. Further, suppose that $*$ is homogeneous.

Choose a \mathbb{K}^\times -invariant point $\chi \in \text{Spec}(A)$. We have the natural structure of a quantum algebra on $\mathcal{A}_h^\wedge := A_h^\wedge[[\hbar]]$. We will be particularly interested in $\mathcal{A}_h = \mathcal{U}_h, \mathcal{W}_h$ (equipped with the Kazhdan \mathbb{K}^\times -actions) with $\chi = (e, \cdot)$.

Let $I_{\chi, h}^\wedge$ denote the inverse image of the maximal ideal $\mathfrak{m}_\chi \subset A_h^\wedge$ of χ in \mathcal{A}_h^\wedge and $I_{\chi, h} := I_{\chi, h}^\wedge \cap \mathcal{A}_h$. Then $I_{\chi, h}^\wedge, I_{\chi, h}$ are two-sided ideals of the corresponding quantum algebras and their powers $(I_{\chi, h}^\wedge)^m, I_{\chi, h}^m$ with respect to the star-products coincide with the powers with respect to the commutative products. The last claim follows easily from (*3). Now it is very easy to see that \mathcal{A}_h^\wedge is naturally isomorphic to the completion $\varprojlim_k \mathcal{A}_h / I_{\chi, h}^k$. If a group Q acts on A preserving χ and $*$, then we have a natural action of \hat{Q} on \mathcal{A}_h^\wedge .

Let \mathcal{M}_h be a finitely generated \mathcal{A}_h -module. To \mathcal{M}_h one can assign its completion $\mathcal{M}_h^\wedge := \varprojlim_k \mathcal{M}_h / I_{\chi, h}^k \mathcal{M}_h$, which has a natural structure of a \mathcal{A}_h^\wedge -module. If \mathcal{M}_h is \mathbb{K}^\times -weakly equivariant, then so is \mathcal{M}_h^\wedge .

Proposition 2.4.1. (1) $\mathcal{M}_h^\wedge = \mathcal{A}_h^\wedge \otimes_{\mathcal{A}_h} \mathcal{M}_h$ and the functor $\mathcal{M}_h \mapsto \mathcal{M}_h^\wedge$ is exact.

(2) $\mathcal{M}_h^\wedge = 0$ if and only if $\chi \notin V(\mathcal{M}_h)$.

(3) If \mathcal{M}_h is $\mathbb{K}[\hbar]$ -flat, then \mathcal{M}_h^\wedge is $\mathbb{K}[[\hbar]]$ -flat.

(4) $\mathcal{M}_h^\wedge / \hbar \mathcal{M}_h^\wedge$ coincides with the completion $(\mathcal{M}_h / \hbar \mathcal{M}_h)_\chi^\wedge$ of the $\mathcal{A}_h / (\hbar)$ -module $\mathcal{M}_h / \hbar \mathcal{M}_h$ at χ .

The proof of this proposition involves a standard machinery with blow-up algebras, compare with [E], Chapter 7.

For an associative algebra \mathcal{A} and a two-sided ideal $\mathcal{J} \subset \mathcal{A}$ one can form the blow-up algebra $\text{Bl}_\mathcal{J}(\mathcal{A}) = \bigoplus_{i=0}^\infty \mathcal{J}^i$. This algebra is positively graded. To ensure nice properties of the completion $\varprojlim_i \mathcal{A} / \mathcal{J}^i$ we need to make sure that the blow-up algebra $\text{Bl}_\mathcal{J}(\mathcal{A})$ is Noetherian.

Lemma 2.4.2. *Let \mathcal{A} be a $\mathbb{K}[\hbar]$ -algebra and \mathcal{J} be a two-sided ideal of \mathcal{A} containing \hbar . Suppose that \mathcal{A} is complete in the \hbar -adic topology. Further, suppose that the algebra $\mathcal{A}/(\hbar)$ is commutative and Noetherian. Finally, suppose that $[\mathcal{J}, \mathcal{J}] \subset \hbar \mathcal{J}$. Then the algebra $\text{Bl}_\mathcal{J}(\mathcal{A})$ is Noetherian.*

Proof. Consider the completed blow-up algebra $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) := \prod_{i=0}^\infty \mathcal{J}^i$. Let \hbar, \hbar' denote the images of $\hbar \in \mathcal{A}$ under the embeddings $\mathcal{A}, \mathcal{J} \hookrightarrow \text{Bl}_\mathcal{J}(\mathcal{A})$. Since \mathcal{A} is complete in the \hbar -adic topology, we see that $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A})$ is complete in the (\hbar, \hbar') -adic topology.

Consider the algebra $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar') = \prod_{i=0}^\infty \mathcal{J}^i / \hbar \mathcal{J}^{i-1}$ (here we assume that $\mathcal{J}^{-1} = \mathcal{J}^0 = \mathcal{A}$). It has a decreasing filtration $F_i \hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar') := \prod_{j \geq i} \mathcal{J}^j / \hbar \mathcal{J}^{j-1}$. The associated graded algebra is nothing else but $\bigoplus_{i=0}^\infty \mathcal{J}^i / \hbar \mathcal{J}^{i-1} = \text{Bl}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$. Let us show that the last algebra is commutative and Noetherian.

Commutativity of $\text{Bl}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ means $[\mathcal{J}^i, \mathcal{J}^j] \subset \hbar \mathcal{J}^{i+j-1}$. This follows easily from $[\mathcal{J}, \mathcal{J}] \subset \hbar \mathcal{J}$. The algebra $\text{Bl}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ is commutative and generated by $\mathcal{J} / (\hbar)$ over the Noetherian algebra $\mathcal{A}_h / (\hbar)$. It follows that $\text{Bl}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ is Noetherian.

Since $\text{Bl}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ is the associated graded of $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ with respect to the complete separated filtration, we see that $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ is Noetherian. Since $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A})$ is complete in the (\hbar, \hbar') -adic topology, and the quotient $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A}) / (\hbar, \hbar')$ is Noetherian, we see that $\hat{\text{Bl}}_\mathcal{J}(\mathcal{A})$ itself is Noetherian.

Let us show that the Noetherian property for $\hat{\text{Bl}}_{\mathcal{J}}(\mathcal{A})$ implies that for $\text{Bl}_{\mathcal{J}}(\mathcal{A})$. More generally, let $B := \bigoplus_{i \geq 0} B_i$ be a $\mathbb{Z}_{\geq 0}$ -graded algebra and let \hat{B} be the completion with respect to this grading (e.g., $B = \text{Bl}_{\mathcal{J}}(\mathcal{A})$, $\hat{B} = \hat{\text{Bl}}_{\mathcal{J}}(\mathcal{A})$). Suppose that \hat{B} is Noetherian. We want to prove that B is also Noetherian.

Consider the algebra $\hat{B}[\hbar]$. It is Noetherian. We have a \mathbb{K}^\times -action on $\hat{B}[\hbar]$ defined as follows: the action on \hat{B} is induced from the grading on B , while $t \cdot \hbar := t^{-1} \hbar$ for any $t \in \mathbb{K}^\times$. Consider the embedding $B \rightarrow \hat{B}[\hbar]$ sending $b \in B_i$ to $b \hbar^i$. This embedding gives an identification $B \cong \hat{B}[\hbar]^{\mathbb{K}^\times}$.

Pick a left ideal $I \subset B = \hat{B}[\hbar]^{\mathbb{K}^\times}$. The left $\hat{B}[\hbar]$ -ideal $\hat{B}[\hbar]I$ is generated by elements $j_1, \dots, j_k \in I$. We claim that j_1, \dots, j_k generate the left ideal $I \subset B$. Indeed, let $j \in I$ and let $b_1, \dots, b_k \in \hat{B}[\hbar]$ be such that $j = \sum_{i=1}^k b_i j_i$. Write $b_i := \sum_{l=0}^{d_i} b_{il} \hbar^l$ and $b_{il} := \sum_{q \geq 0} b_{il}^q \hbar^q$ with $b_{il}^q \in B_q$. Then $b'_i := \sum_{l=0}^{d_i} b_{il}^l \hbar^l$ lies in $\hat{B}[\hbar]^{\mathbb{K}^\times}$ and $j = \sum_{i=1}^k b'_i j_i$. \square

Proof of Proposition 2.4.1. Let us prove the first claim.

First of all, we remark that the algebra \mathcal{A}_\hbar is Noetherian (this can be proved using the standard argument of Hilbert, since $\mathcal{A}_\hbar = A[\hbar]$ as a vector space, $\mathcal{A}_\hbar/(\hbar) = A$, and A is Noetherian).

Consider the completion \mathcal{A}'_\hbar of \mathcal{A}_\hbar in the \hbar -adic topology. To any (left) finitely generated \mathcal{A}_\hbar -module \mathcal{M}_\hbar one can assign its completion $\mathcal{M}'_\hbar := \varprojlim \mathcal{M}_\hbar / \hbar^k \mathcal{M}_\hbar$, which has a natural structure of a \mathcal{A}'_\hbar -module. The blow-up algebra $\text{Bl}_{(\hbar)}(\mathcal{A}_\hbar)$ is nothing else but the polynomial algebra $\mathcal{A}_\hbar[\hbar]$ and, in particular, is Noetherian. So applying the argument of [E], Chapter 7, we see that

- (1) the functor of the \hbar -adic completion is exact.
- (2) $\mathcal{M}'_\hbar = \mathcal{A}'_\hbar \otimes_{\mathcal{A}_\hbar} \mathcal{M}_\hbar$.

Let $I'_{\chi, \hbar}$ be the completion of $I_{\chi, \hbar}$ in the \hbar -adic topology. Lemma 2.4.2 applies to $\mathcal{A} := \mathcal{A}'_\hbar$ and $\mathcal{J} := I'_{\chi, \hbar}$ because $[\mathcal{J}, \mathcal{J}] \subset \hbar^2 \mathcal{A}$. So $\text{Bl}_{\mathcal{J}}(\mathcal{A})$ is Noetherian. It follows that the Artin-Rees lemma (see, for example, [E], Chapter 5) holds for $\mathcal{J} \subset \mathcal{A}$.

Following the proof in the commutative case that can be found in [E], chapter 7, we prove assertion (1). Assertions (3) and (4) follow from the exactness of $0 \rightarrow \mathcal{M}_\hbar^\wedge \xrightarrow{\hbar} \mathcal{M}_\hbar^\wedge \rightarrow (\mathcal{M}_\hbar / \hbar \mathcal{M}_\hbar)^\wedge_\chi$, which stems from (1). Assertion (2) follows from (4). \square

In the sequel we will need the following corollary of Proposition 2.4.1.

Corollary 2.4.3. *Let \mathcal{I}_\hbar be a right ideal in \mathcal{A}_\hbar , \mathcal{M}_\hbar be an \mathcal{A}_\hbar -bimodule that is finitely generated as a left \mathcal{A}_\hbar -module. Let $\underline{\mathcal{M}}_\hbar$ be the annihilator (of the right action) of \mathcal{I}_\hbar in \mathcal{M}_\hbar . Then the annihilator of \mathcal{I}_\hbar in \mathcal{M}_\hbar^\wedge coincides with $\underline{\mathcal{M}}_\hbar^\wedge$.*

Proof. At first, consider the case where \mathcal{I}_\hbar is generated by a single element, say a . Consider the exact sequence $0 \rightarrow \underline{\mathcal{M}}_\hbar \rightarrow \mathcal{M}_\hbar \xrightarrow{a} \mathcal{M}_\hbar$. By assertion (1) of Proposition 2.4.1, the completion functor is exact. The same assertion implies that the completion functor is \mathcal{A}_\hbar^{op} -linear. So we get the exact sequence $0 \rightarrow \underline{\mathcal{M}}_\hbar^\wedge \rightarrow \mathcal{M}_\hbar^\wedge \xrightarrow{a} \mathcal{M}_\hbar^\wedge$. This completes the proof in the case when \mathcal{I}_\hbar is generated by one element.

Let us proceed to the general case. The ideal \mathcal{I}_\hbar is generated by some elements a_1, \dots, a_k . Let $\underline{\mathcal{M}}_\hbar^i$ stand for the right annihilator of a_i in \mathcal{M}_\hbar . Then $\underline{\mathcal{M}}_\hbar = \bigcap_i \underline{\mathcal{M}}_\hbar^i$. Since the completion functor is exact, we see that $\underline{\mathcal{M}}_\hbar^\wedge = \bigcap_i \underline{\mathcal{M}}_\hbar^{i^\wedge}$. To complete the proof apply the result of the previous paragraph. \square

Lemma 2.4.4. (1) *The algebra \mathcal{A}_\hbar^\wedge is Noetherian.*

- (2) Any finitely generated left \mathcal{A}_h^\wedge -module is complete and separated with respect to $I_{\chi, h}^\wedge$ -adic topology.
- (3) Any submodule in a finitely generated \mathcal{A}_h^\wedge -module is closed with respect to $I_{\chi, h}^\wedge$ -adic topology.

Proof. Assertion (1) is easy, for example, it follows from the observation that \mathcal{A}_h^\wedge is complete in the \hbar -adic topology, and $\mathcal{A}_h^\wedge/(\hbar) = A_\chi^\wedge$ is Noetherian. To prove (2) and (3) we notice that Lemma 2.4.2 applies to $\mathcal{A} := \mathcal{A}_h^\wedge$, $\mathcal{J} := I_{\chi, h}^\wedge$. So the Artin-Rees lemma holds for $\mathcal{J} \subset \mathcal{A}$. (3) and the claim in (2) that the filtration is complete are direct corollaries of the Artin-Rees lemma. The claim in (2) that the filtration is separated is proved in the same way as the Krull separation theorem, compare with [E], Section 5.3. \square

2.5. Harish-Chandra bimodules over quantum algebras. In this subsection we will introduce categories of Harish-Chandra bimodules for the quantum algebras $\mathcal{U}_h, \mathcal{W}_h, \mathcal{U}_h^\wedge, \mathcal{W}_h^\wedge$ and their Q -equivariant analogs.

Equip \mathcal{U}_h with the "doubled" usual \mathbb{K}^\times -action ($t.\xi = t^2\xi, t.\hbar = t\hbar, t \in \mathbb{K}^\times, \xi \in \mathfrak{g}$) and \mathcal{W}_h with the Kazhdan \mathbb{K}^\times -action. For $\mathcal{A}_h = \mathcal{U}_h$ or \mathcal{W}_h we say that a graded \mathcal{A}_h -bimodule \mathcal{M}_h , where the left and the right actions of $\mathbb{K}[\hbar]$ coincide, is Harish-Chandra if

- (i) \mathcal{M}_h is $\mathbb{K}[\hbar]$ -flat.
- (ii) \mathcal{M}_h is finitely generated as a \mathcal{A}_h -bimodule.
- (iii) $[a, m] \in \hbar^2 \mathcal{M}_h$ for any $a \in \mathcal{A}_h, m \in \mathcal{M}_h$.

The following lemma describes simplest properties of Harish-Chandra bimodules.

Lemma 2.5.1. (1) \mathcal{M}_h is finitely generated both as a left and as a right \mathcal{A}_h -module.
 (2) All graded components of \mathcal{M}_h are finite dimensional.
 (3) For $\mathcal{A}_h = \mathcal{U}_h$ the adjoint action of \mathfrak{g} on \mathcal{M}_h : $(\xi, m) \mapsto \frac{1}{\hbar^2}[\xi, m], \xi \in \mathfrak{g}, m \in \mathcal{M}_h$, is locally finite.

Proof. Let m_1, \dots, m_k be homogeneous generators of the \mathcal{A}_h -bimodule \mathcal{M}_h and let $\underline{\mathcal{M}}_h$ be the left submodule in \mathcal{M}_h generated by m_1, \dots, m_k . From (iii) it follows that $\mathcal{M}_h = \underline{\mathcal{M}}_h + \hbar \mathcal{M}_h$. But \mathcal{A}_h positively graded. So (ii) implies that the grading on \mathcal{M}_h is bounded from below. Now the proof of (1) follows easily.

(2) follows from (ii).

To prove (3) we note that the map $\mathcal{M}_h \rightarrow \mathcal{M}_h, m \mapsto \frac{1}{\hbar^2}[\xi, m], \xi \in \mathfrak{g}$, is homogeneous. \square

Let $\mathcal{M} \in \text{HC}(\mathcal{U})$. Slightly modifying a standard definition, we say that a filtration $F_i \mathcal{M}$ is said to be good if it is $\text{ad}(\mathfrak{g})$ -stable, compatible with the "doubled" standard filtration $F_i \mathcal{U} := F_{[i/2]}^{st} \mathcal{U}$ on \mathcal{U} , and $\text{gr } \mathcal{M}$ is a finitely generated $\text{gr } \mathcal{U} = S(\mathfrak{g})$ -module. To construct a good filtration take a $\text{ad}(\mathfrak{g})$ -stable finite dimensional subspace $M^0 \subset \mathcal{M}$ generating \mathcal{M} as a bimodule, and set $F_i \mathcal{M} := F_i \mathcal{U} M_0$.

Given a good filtration $F_i \mathcal{M}$ on $\text{HC}(\mathcal{U})$ form the Rees $\mathcal{U}_h = R_h(\mathcal{U})$ -bimodule $R_h(\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \hbar^i F_i \mathcal{M}$. Then $R_h(\mathcal{M})$ becomes an object in $\text{HC}(\mathcal{U}_h)$, the i -th graded component being $\hbar^i F_i \mathcal{M}$. Conversely, for $\mathcal{M}_h \in \text{HC}(\mathcal{U}_h)$ the quotient $\mathcal{M}_h/(\hbar - 1)\mathcal{M}_h$ lies in $\text{HC}(\mathcal{U})$ and the filtration induced by the grading on \mathcal{M}_h is good. It is clear that the assignments $\mathcal{M} \mapsto R_h(\mathcal{M}), \mathcal{M}_h \mapsto \mathcal{M}_h/(\hbar - 1)\mathcal{M}_h$ are inverse to each other.

Now let \mathcal{N} be a \mathcal{W} -bimodule. We say that \mathcal{N} is *Harish-Chandra* if there is a filtration $F_i \mathcal{N}$ on \mathcal{N} such that $R_h(\mathcal{N}) \in \text{HC}(\mathcal{W}_h)$, equivalently, $\text{gr } \mathcal{N}$ is a finitely generated $\mathbb{K}[S]$ -module, and the filtration $F_i \mathcal{N}$ is almost commutative in the sense that $[K_i \mathcal{W}, F_j \mathcal{N}] \subset F_{i+j-2} \mathcal{N}$.

Recall the subgroup $Q := Z_G(e, h, f) \subset G$. Now let us define Q -equivariant Harish-Chandra \mathcal{W}_h -bimodules. We say that a HC \mathcal{W}_h -bimodule \mathcal{N}_h is Q -equivariant if it is equipped with a Q -action such that

- (iQ) The Q -action preserves the grading.
- (iiQ) The structure map $\mathcal{W}_h \otimes \mathcal{N}_h \otimes \mathcal{W}_h \rightarrow \mathcal{N}_h$ is Q -equivariant.
- (iiiQ) The differential of the Q -action on \mathcal{N}_h coincides with the the action of $\mathfrak{q} \hookrightarrow \mathcal{W}_h$ given by $\frac{1}{h^2}[\xi, \cdot], \xi \in \mathfrak{q}$.

Analogously we define the category $\mathrm{HC}^Q(\mathcal{W})$ of Q -equivariant Harish-Chandra bimodules.

Now let us introduce suitable categories for the completed algebras $\mathcal{U}_h^\wedge, \mathcal{W}_h^\wedge$. Consider the Kazhdan actions of \mathbb{K}^\times on $\mathcal{A}_h^\wedge := \mathcal{U}_h^\wedge$ or \mathcal{W}_h^\wedge .

We say that a \mathbb{K}^\times -weakly equivariant \mathcal{A}_h^\wedge -bimodule \mathcal{M}'_h , where the left and the right actions of $\mathbb{K}[[\hbar]]$ coincide, is Harish-Chandra if

- (i $^\wedge$) \mathcal{M}'_h is $\mathbb{K}[[\hbar]]$ -flat.
- (ii $^\wedge$) \mathcal{M}'_h is a finitely generated \mathcal{A}_h^\wedge -bimodule and complete in the $I_{\chi, h}^\wedge$ -adic topology.
- (iii $^\wedge$) $[a, m] \in \hbar^2 \mathcal{M}'_h$ for any $a \in \mathcal{A}_h^\wedge, m \in \mathcal{M}'_h$.

We remark that (ii $^\wedge$) and (iii $^\wedge$) easily imply that \mathcal{M}'_h is finitely generated both as a left and as a right \mathcal{A}_h^\wedge . Conversely, any finitely generated left \mathcal{A}_h^\wedge -bimodule is complete in the $I_{\chi, h}^\wedge$ -adic topology, see Lemma 2.4.4. The category of Harish-Chandra \mathcal{A}_h^\wedge -bimodules will be denoted by $\mathrm{HC}(\mathcal{A}_h^\wedge)$.

The definition of a Q -equivariant Harish-Chandra \mathcal{A}_h^\wedge is given by analogy with that of a Q -equivariant Harish-Chandra \mathcal{W}_h -bimodule (one should replace (iQ) with the condition that the Q -action commutes with the \mathbb{K}^\times -action). The category of Q -equivariant HC \mathcal{A}_h^\wedge -bimodules is denoted by $\mathrm{HC}^Q(\mathcal{A}_h^\wedge)$.

We remark that the categories $\mathrm{HC}(\mathcal{U}_h), \mathrm{HC}^Q(\mathcal{U}_h^\wedge)$ etc. are $\mathbb{K}[[\hbar]]$ -linear but not abelian (due to the flatness condition, cokernels are undefined in general). It still makes sense to speak about exact sequences in our categories. Also the categories in consideration have tensor product functors: for instance, for $\mathcal{M}_h^1, \mathcal{M}_h^2 \in \mathrm{HC}(\mathcal{A}_h^\wedge)$, one can take the usual tensor product $\mathcal{M}_h \otimes_{\mathcal{A}_h^\wedge} \mathcal{N}_h$ of bimodules and then take its quotient by the \hbar -torsion. So this tensor product satisfies (i $^\wedge$). Clearly, it satisfies (iii $^\wedge$). To see that it satisfies (ii $^\wedge$) we remark that $\mathcal{M}_h^1 \otimes_{\mathcal{A}_h^\wedge} \mathcal{M}_h^2$ is finitely generated as a left \mathcal{A}_h^\wedge -module, because $\mathcal{M}_h^1, \mathcal{M}_h^2$ are.

The categories $\mathrm{HC}(\mathcal{U}_h)$ and $\mathrm{HC}^Q(\mathcal{U}_h^\wedge)$ are related via the completion functor as explained in the following lemma.

Lemma 2.5.2. *Let $\mathcal{M}_h \in \mathrm{HC}(\mathcal{U}_h)$.*

- (1) *The completion $\mathcal{M}_h^\wedge := \varprojlim \mathcal{M}_h / I_{\chi, h}^k \mathcal{M}_h$ has a natural structure of a Q -equivariant $\mathrm{HC} \mathcal{U}_h^\wedge$ -bimodule.*
- (2) *The completion functor $\mathrm{HC}(\mathcal{U}_h) \rightarrow \mathrm{HC}^Q(\mathcal{U}_h^\wedge)$ is exact and tensor.*

Proof. To see that \mathcal{M}_h^\wedge is indeed a \mathcal{U}_h^\wedge -bimodule we remark that $I_{\chi, h}^k \mathcal{M}_h = \mathcal{M}_h I_{\chi, h}^k$ thanks to (iii). Equip \mathcal{M}_h with a Kazhdan \mathbb{K}^\times -action: $(t, m) \mapsto t^{2i} \gamma(t) m$ for m of degree i and with a Q -action restricted from the G -action (the latter is integrated from the adjoint \mathfrak{g} -action, $\xi \mapsto \frac{1}{h^2} \mathrm{ad}(\xi)$). It is straightforward to verify that \mathcal{M}_h^\wedge becomes an object of $\mathrm{HC}^Q(\mathcal{U}_h^\wedge)$.

(2) follows from assertion (1) of Proposition 2.4.1. \square

Similarly, we have a completion functor $\mathrm{HC}^Q(\mathcal{W}_h) \rightarrow \mathrm{HC}^Q(\mathcal{W}_h^\wedge)$.

3. CONSTRUCTION OF FUNCTORS

3.1. Correspondence between ideals. Here we recall the construction of mappings between the sets $\mathfrak{Id}(\mathcal{W}), \mathfrak{Id}(\mathcal{U})$, see [Lo1], Subsection 3.4 for details and proofs.

Recall the algebras $\mathcal{U}_h, \mathcal{U}_h^\wedge, \mathcal{W}_h, \mathcal{W}_h^\wedge, \mathbf{A}_h, \mathbf{A}_h^\wedge, \mathbf{A}_h^\wedge(\mathcal{W}_h)$ and the isomorphism $\Phi_h : \mathcal{U}_h^\wedge \rightarrow \mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge)$ established in Subsection 2.3.

The map $\mathcal{I} \mapsto \mathcal{I}^\dagger : \mathfrak{Id}(\mathcal{W}) \rightarrow \mathfrak{Id}(\mathcal{U})$ is constructed as follows. Construct the ideal $\mathcal{I}_h := R_h(\mathcal{I}) \subset R_h(\mathcal{W}) = \mathcal{W}_h$ and take its completion $\mathcal{I}_h^\wedge \subset \mathcal{W}_h^\wedge$. Then construct the ideal $\mathbf{A}_h^\wedge(\mathcal{I}_h^\wedge) := \mathbf{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[h]]} \mathcal{I}_h^\wedge$ in $\mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge) = \mathcal{U}_h^\wedge$. Taking its intersection with $\mathcal{U}_h \subset \mathcal{U}_h^\wedge$, we get an ideal $\mathcal{I}_h^\dagger \subset \mathcal{U}_h$. Finally, set $\mathcal{I}^\dagger := \mathcal{I}_h^\dagger / (\hbar - 1)$. We remark that, by construction, the map $\mathcal{I} \mapsto \mathcal{I}^\dagger$ is Q -invariant: $(q\mathcal{I})^\dagger = \mathcal{I}^\dagger$ for any $q \in Q$.

To construct a map $\mathcal{J} \mapsto \mathcal{J}_\dagger : \mathfrak{Id}(\mathcal{U}) \rightarrow \mathfrak{Id}(\mathcal{W})$ we, first, pass from \mathcal{J} to $\mathcal{J}_h := R_h(\mathcal{J}) \subset \mathcal{U}_h$ and then to its completion $\mathcal{J}_h^\wedge \subset \mathcal{U}_h^\wedge = \mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge)$. The completion is \hbar -saturated has the form $\mathbf{A}_h^\wedge(\mathcal{I}_h^\wedge)$ for a unique \mathbb{K}^\times -stable ideal \mathcal{I}_h^\wedge . Then take the intersection $\mathcal{I}_h := \mathcal{J}_h^\wedge \cap \mathcal{W}_h$ (\mathcal{I}_h is indeed dense in \mathcal{J}_h^\wedge) and, finally, set $\mathcal{J}_\dagger := \mathcal{I}_h / (\hbar - 1)$. The ideal \mathcal{J}_\dagger is Q -stable for any \mathcal{J} .

These two maps enjoy the following properties ([Lo1], Theorem 1.2.2 and its proof in Subsection 3.4).

- Theorem 3.1.1.**
- (i) $(\mathcal{I}_1 \cap \mathcal{I}_2)^\dagger = \mathcal{I}_1^\dagger \cap \mathcal{I}_2^\dagger$.
 - (ii) $\mathcal{I} \supset (\mathcal{I}^\dagger)_\dagger$ and $\mathcal{J} \subset (\mathcal{J}_\dagger)^\dagger$ for any $\mathcal{I} \in \mathfrak{Id}(\mathcal{W}), \mathcal{J} \in \mathfrak{Id}(\mathcal{U})$.
 - (iii) $\mathcal{I}^\dagger \cap \mathcal{Z}(\mathfrak{g}) = \mathcal{I} \cap \mathcal{Z}(\mathfrak{g})$. In the r.h.s. $\mathcal{Z}(\mathfrak{g})$ is embedded into \mathcal{W} as explained in Subsection 2.2.
 - (iv) \mathcal{I}^\dagger is primitive provided \mathcal{I} is.
 - (v) For any primitive $\mathcal{J} \in \mathfrak{Id}_\mathbb{O}(\mathcal{U})$ we have $\{\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W}) \mid \mathcal{I}^\dagger = \mathcal{J}\} = \{\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W}) \mid \mathcal{J}_\dagger \subset \mathcal{I}\}$.
 - (vi) $\text{codim}_{\mathcal{W}} \mathcal{J}_\dagger = \text{mult}_{\overline{\mathbb{O}}} \mathcal{U} / \mathcal{J}$ provided $\overline{\mathbb{O}}$ is an irreducible component of $V(\mathcal{U} / \mathcal{J})$.
 - (vii) Let $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$ be primitive. Then $\text{Grk}(\mathcal{U} / \mathcal{I}^\dagger) \leq \text{Grk}(\mathcal{W} / \mathcal{I}) = (\dim \mathcal{W} / \mathcal{I})^{1/2}$. Here Grk stands for the Goldie rank.

Remark 3.1.2. Actually, $\mathcal{J}_\dagger \cap \mathcal{Z}(\mathfrak{g}) \supset \mathcal{J} \cap \mathcal{Z}(\mathfrak{g})$. This can be proved either using an alternative description of \mathcal{J}_\dagger given in Subsection 3.5 or deduced directly from the definition above.

Remark 3.1.3. Actually, one can show that $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$ implies $\mathcal{I}^\dagger \in \mathfrak{Id}_\mathbb{O}(\mathcal{U})$. Indeed, we have seen in [Lo1] that this holds provided \mathcal{I} is primitive. The proof is based on the Joseph irreducibility theorem: $V(\mathcal{U} / \mathcal{J})$ is irreducible for any primitive ideal $\mathcal{J} \subset \mathcal{U}$. Let us deduce the assertion for an arbitrary $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$. Assertion (i) of Theorem 3.1.1 shows that it holds for all semiprime ideals from $\mathfrak{Id}_{fin}(\mathcal{W})$. Tracking the construction of $\mathcal{I} \mapsto \mathcal{I}^\dagger$ one can see that $(\mathcal{I}^\dagger)^k \subset (\mathcal{I}^k)^\dagger$. This yields the assertion in the general case.

However, it is possible to prove that $\mathcal{I}^\dagger \in \mathfrak{Id}_\mathbb{O}(\mathcal{U})$ for $\mathcal{I} \in \mathfrak{Id}_{fin}(\mathcal{W})$ without referring to the Joseph theorem and deduce the latter from here. We will make a remark about this, Remark 3.4.4. The only part of Theorem 3.1.1 used in Remark 3.4.4 is (ii), which is pretty straightforward from the constructions.

3.2. Homogeneous vector bundles. In this subsection we will establish category equivalences between various ramifications of the category of homogeneous vector bundles. Namely, let G be as above and H be a subgroup of G such that

- (A) G/H is quasi-affine (H is *observable* in the terminology of [Gr]).
- (B) $\mathbb{K}[G/H]$ is finitely generated.

Consider the category $\mathrm{HVB}_{G/H}$ of homogeneous vector bundles on G/H , i.e., of G -equivariant coherent sheaves on G/H .

Lemma 3.2.1. (1) *If $H \subset G$ satisfies (A),(B), then H° does.*

(2) *If $H \subset G$ satisfies (A),(B), then $\Gamma(M)$ is a finitely generated $\mathbb{K}[G/H]$ -module for any $M \in \mathrm{HVB}_{G/H}$.*

(3) *For any $x \in \mathfrak{g} \cong \mathfrak{g}^*$ the stabilizer G_x satisfies (A),(B).*

Proof. (1): For H° (A) follows from [Gr], Corollary 2.3, and (B) follows from [Gr], Theorem 4.1.

(2): This follows from [Gr], Lemma 23.1(h).

(3): This follows [Gr], Theorem 4.3, since \overline{Gx} consists of finitely many orbits of even dimension. \square

We need the following the following categories related to $\mathrm{HVB}_{G/H}$. First of all, we consider the category Mod_H of finite dimensional H -modules.

To construct the second category fix an affine G -variety X such that there is an open G -equivariant embedding $G/H \hookrightarrow X$ with $\mathrm{codim}_X X \setminus (G/H) \geq 2$. We remark that $\mathbb{K}[G/H]$ is the normalization of $\mathbb{K}[X]$. Then let $\mathrm{Coh}^G(X)$ denote the category of G -equivariant coherent sheaves on X . This category has the Serre subcategory consisting of all modules supported on $X \setminus G/H$. The quotient category will be denoted by $\mathrm{HVB}_{G/H}^X$. Finally, fix an H -stable point $x \in G/H$. Consider the category $\mathrm{HVB}_{G/H}^\wedge$ consisting of all finitely generated $\mathbb{K}[G/H]_x^\wedge$ -modules M equipped additionally by actions of \mathfrak{g} and H subject to the following compatibility conditions:

- (a) The action map $\mathbb{K}[G/H]_x^\wedge \otimes M \rightarrow M$ is \mathfrak{g} - and H -equivariant.
- (b) The action map $\mathfrak{g} \otimes M \rightarrow M$ is H -equivariant.
- (c) The differential of the H -action on M coincides with the restriction of the \mathfrak{g} -action to \mathfrak{h} .

We have various functors between the categories in interest. For instance, to $P \in \mathrm{Mod}_H$ we can assign the homogeneous vector bundle $\mathcal{F}_1(P) := G *_H P$ on G/H with fiber P , see, for instance, [PV], Section 4.8, for details. It is clear that the functor \mathcal{F}_1 is an equivalence, a quasi-inverse equivalence \mathcal{F}_1^{-1} is given by taking the fiber at x . Further, we have the completion functor $\mathcal{F}_2 : \mathrm{HVB}_{G/H} \rightarrow \mathrm{HVB}_{G/H}^\wedge$. The H -action on the completion comes from the H -action on $G *_H P$ restricted from the G -action. It is given by the equality $h.(g *_H v) = hgh^{-1} *_H hv$, $g \in G, h \in H, v \in P$.

Then we can consider the functor $\mathcal{F}_3 : \mathrm{HVB}_{G/H}^\wedge \rightarrow \mathrm{Mod}_H, M \mapsto M/\mathfrak{m}_x M$, where \mathfrak{m}_x denote the maximal ideal in $\mathbb{K}[G/H]_x^\wedge$. It is clear that $\mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1 = \mathrm{id}$. Finally, we have the functor $\tilde{\mathcal{F}}_4 : \mathrm{Coh}^G(X) \rightarrow \mathrm{HVB}_{G/H}^X$ of restriction to G/H . It induces the functor $\mathcal{F}_4 : \mathrm{HVB}_{G/H}^X \rightarrow \mathrm{HVB}_{G/H}$.

Proposition 3.2.2. *The functors $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ are equivalences.*

Proof. Let us show that \mathcal{F}_4 is an equivalence. Thanks to assertion (2) of Lemma 3.2.1, $\Gamma(M)$ is a finitely generated $\mathbb{K}[G/H]$ - and hence also a $\mathbb{K}[X]$ -module. So we can consider $\Gamma(M)$ as an element of $\mathrm{Coh}^G(X)$. Let $\mathcal{F}_4'(M)$ be the image of $\Gamma(M)$ in $\mathrm{HVB}_{G/H}^X$. It is clear that $\Gamma(\bullet)$ is right adjoint to $\tilde{\mathcal{F}}_4$. Since any M can be extended to at least some coherent sheaf on X and X is affine, we see that the fiber of $\Gamma(M)$ in x is the same as that of M . This observation implies that \mathcal{F}_4' is a two-sided inverse to \mathcal{F}_4 .

Now let us show that \mathcal{F}_3 is an equivalence. For this it is enough to show that $\mathcal{F}_2 \circ \mathcal{F}_1 \circ \mathcal{F}_3$ is isomorphic to the identity functor.

Pick $M \in \text{HVB}_{G/H}^\wedge$ and let $P = \mathcal{F}_3(M) = M/\mathfrak{m}_x M$. Let $\pi : M \rightarrow P$ denote the projection. Recall that we have a \mathfrak{g} - and hence a \mathcal{U} -action on M . Map a pair $(u, m), u \in \mathcal{U}, m \in M$, to $\pi(um)$. This is an H -equivariant bilinear map, in particular, $\pi(\xi um) = \xi \pi(um)$ for any $\xi \in \mathfrak{h}$. In other words, we get a linear map $\iota : M \rightarrow \text{Hom}_{\mathfrak{h}}(\mathcal{U}, P)$, where the latter stands for the space of all \mathfrak{h} -equivariant (as above) linear maps $\mathcal{U} \rightarrow P$. The map ι is \mathfrak{g} - and H -equivariant, where $\eta \in \mathfrak{g}, h \in H$ act on $\varphi : \mathcal{U} \rightarrow P$ by $[\eta \cdot \varphi](u) = \varphi(u\eta), [h \cdot \varphi](u) = h\varphi(h^{-1} \cdot u)$ (it is straightforward to check that the actions are well-defined).

Now we claim that ι is an isomorphism. Let us check that ι is injective. Assume the converse, then there is $m \neq 0$ with $\pi(um) = 0$ for all $u \in \mathcal{U}$. But \mathfrak{g} spans the $\mathbb{K}[G/H]_x^\wedge$ -module $\text{Der}(\mathbb{K}[G/H]_x^\wedge)$. It follows that $m \in \bigcap_n \mathfrak{m}_x^n M$. Since M is finitely generated, the latter intersection is zero by the Krull separation theorem.

Let us check that ι is surjective. First of all, there is a natural projection $\rho : \text{Hom}_{\mathfrak{h}}(\mathcal{U}, P) \rightarrow P$, $\rho(\varphi) = \varphi(1)$. It is straightforward to see that $\pi = \rho \circ \iota$. On the other hand, $\text{im } \iota$ is a \mathcal{U} -submodule in $\text{Hom}_{\mathfrak{h}}(\mathcal{U}, P)$. But $\ker \rho$ contains no nonzero \mathcal{U} -submodules: indeed an element φ in such a submodule satisfies $0 = \rho(u \cdot \varphi) = \varphi(u)$ for any $u \in \mathcal{U}$. So we see that ι is an isomorphism.

In particular, we can identify $\mathbb{K}[G/H]_x^\wedge$ with $\text{Hom}_{\mathfrak{h}}(\mathcal{U}, \mathbb{K})$. The multiplication on $\mathbb{K}[G/H]_x^\wedge$ corresponds to the multiplication on $\text{Hom}_{\mathfrak{h}}(\mathcal{U}, \mathbb{K})$ induced by the comultiplication $\Delta : \mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{U}$. Similarly, $\text{Hom}_{\mathfrak{h}}(\mathcal{U}, P)$ becomes a $\mathbb{K}[G/H]_x^\wedge$ -module. By the Leibniz identity, for $u \in \mathcal{U}, f \in \mathbb{K}[G/H]_x^\wedge, \varphi \in M$ the element $u \cdot (f\varphi)$ coincides with the image of $\Delta(u) \cdot f \otimes \varphi$ under the action map $\mathbb{K}[G/H]_x^\wedge \otimes M \rightarrow M$. It follows ι becomes a homomorphism of $\mathbb{K}[G/H]_x^\wedge$ -modules and hence an isomorphism in $\text{HVB}_{G/H}^\wedge$.

But we also get the identification $\iota : \mathcal{F}_2(\mathcal{F}_1(P)) \rightarrow \text{Hom}_{\mathfrak{h}}(\mathcal{U}, P)$. This gives rise to an isomorphism $M \rightarrow \mathcal{F}_2 \circ \mathcal{F}_1 \circ \mathcal{F}_3(M)$. By the construction, this isomorphism is functorial. \square

We need an alternative description of the equivalence $\mathcal{F}_4 \circ \mathcal{F}_1 \circ \mathcal{F}_3$. This description will be crucial in the sequel.

Namely, pick $M \in \text{HVB}_{G/H}^\wedge$. Consider the space $M_{\mathfrak{g}-l.f.}$ of \mathfrak{g} -l.f. vectors. The subspace $M_{\mathfrak{g}-l.f.} \subset M$ is H -stable, let ρ denote the corresponding representation of H in $M_{\mathfrak{g}-l.f.}$. On the other hand, M has the structure of a G -module integrated from the \mathfrak{g} -action. Restricting the representation of G to H , we get the representation ρ' of H in $M_{\mathfrak{g}-l.f.}$. The action map $\mathfrak{g} \otimes M_{\mathfrak{g}-l.f.} \rightarrow M$ is equivariant with respect to both representations ρ, ρ' . So $\rho(h)\rho'(h)^{-1}$ commutes with \mathfrak{g} for any $h \in H$. The differentials of ρ, ρ' coincide hence $\rho(h) = \rho'(h)$ for any $h \in H^\circ$. So $\sigma(h) = \rho(h)\rho'(h)^{-1}$ defines a representation of H/H° in $M_{\mathfrak{g}-l.f.}$ commuting with G .

Proposition 3.2.3. *Let $P \in \text{Mod}_H$ and $M = \mathcal{F}_2 \circ \mathcal{F}_1(P)$. Then*

- (1) $M_{\mathfrak{g}-l.f.} = \Gamma(G/H^\circ, G *_{H^\circ} P)$,
- (2) $M_{\mathfrak{g}-l.f.}^{H/H^\circ} = \Gamma(G/H, G *_{H^\circ} P) (= \tilde{\mathcal{F}}_4 \circ \mathcal{F}_1(P))$.

Proof. (1): Fix a point $\tilde{x} \in G/H^\circ$ mapping to x under the natural projection $G/H^\circ \rightarrow G/H$

The objects in $\text{HVB}_{G/H^\circ}^\wedge$ and $\text{HVB}_{G/H}^\wedge$ obtained from $P \in \text{Mod}_H$ are naturally isomorphic. So we have a natural map $N := \Gamma(G/H^\circ, G *_{H^\circ} P) \rightarrow M$. This map is injective: any section in its kernel vanishes in x together with its G -translates and hence vanishes everywhere. Since $N = (\mathbb{K}[G] \otimes P)^H \subset \mathbb{K}[G] \otimes P$ and $\mathbb{K}[G] \otimes P$ is clearly \mathfrak{g} -l.f., we see that $N \subset M_{\mathfrak{g}-l.f.}$.

First of all, let us check that $N^G = (M_{\mathfrak{g}-l.f.})^G = M^{\mathfrak{g}}$. Recall that we have the projection $\pi : M \rightarrow P$. Analogously to the proof of the injectivity of ι in Proposition 3.2.2 one gets that the restriction of π to $M^{\mathfrak{g}}$ is injective. Also it is clear that $\pi(M^{\mathfrak{g}}) \subset P^{\mathfrak{h}}$. On the other hand, the restriction of π to N is nothing else but taking the value of a section at \tilde{x} . So $\pi(N^G) = P^{H^\circ} = P^{\mathfrak{h}}$. It follows that $N^G = (M_{\mathfrak{g}-l.f.})^G$.

To prove that $N = M_{\mathfrak{g}-l.f.}$ one needs to verify that the spaces of G -equivariant linear maps $\text{Hom}_G(L, N) \subset \text{Hom}_G(L, M_{\mathfrak{g}-l.f.})$ coincide for an arbitrary irreducible G -module L . But $\text{Hom}_G(L, N) = (L^* \otimes N)^G = \Gamma(G/H^\circ, G *_{H^\circ} (P \otimes L^*))$ and $\text{Hom}_G(L, M_{\mathfrak{g}-l.f.}) = (L^* \otimes M)^{\mathfrak{g}}$. It is clear that $\mathcal{F}_2 \circ \mathcal{F}_1(L^* \otimes P) = L^* \otimes M$. The equality $\text{Hom}_G(L, N) = \text{Hom}_G(L, M_{\mathfrak{g}-l.f.})$ follows from the previous paragraph.

(2): To prove the claim we will need to describe the action σ of H/H° on $M_{\mathfrak{g}-l.f.} = N$. Let us note that H/H° acts on the total space of $G *_{H^\circ} P$ by $h.(g *_{H^\circ} v) = gh^{-1} *_{H^\circ} hv$, $h \in H$. Let us prove that σ is induced by this action.

The G -action on M is induced by the action $g_1.(g *_{H^\circ} v) = g_1g *_{H^\circ} v$. So ρ' is induced from the action $h.(g *_{H^\circ} v) = hg *_{H^\circ} v$. On the other hand, the description of the H -action on $M = \mathcal{F}_2 \circ \mathcal{F}_1(P)$ implies that ρ is induced from the action $h.(g *_{H^\circ} v) = hgh^{-1} *_{H^\circ} hv$. So $\sigma(h) = \rho(h)\rho'(h)^{-1}$ is induced from the required H/H° -action.

Now it remains to notice that $N^{H/H^\circ} = \Gamma(G/H, G *_{H^\circ} P)$. \square

Below we will use results of this subsection in the special case $H := G_\chi, X := \overline{\mathbb{O}}$. The subgroup H enjoys the properties (A),(B) by assertion (3) of Lemma 3.2.1.

3.3. Construction of functors between $\text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h), \text{HC}_{fin}^Q(\mathcal{W}_h)$. In this subsection we construct functors $\bullet_\dagger : \text{HC}(\mathcal{U}_h) \rightarrow \text{HC}^Q(\mathcal{W}_h), \bullet^\dagger : \text{HC}_{fin}^Q(\mathcal{W}_h) \rightarrow \text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$, where the last two categories are defined as follows. $\text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$ is the full subcategory in $\text{HC}(\mathcal{U}_h)$ consisting of all bimodules \mathcal{M}_h with $V(\mathcal{M}_h) \subset \overline{\mathbb{O}}$, while $\text{HC}_{fin}^Q(\mathcal{W})$ is the full subcategory in $\text{HC}^Q(\mathcal{W}_h)$ consisting of all bimodules of finite rank over $\mathbb{K}[[\hbar]]$.

Recall that the algebras \mathcal{U}_h^\wedge and $\mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge)$ are identified by means of the isomorphism Φ_h introduced in the end of Subsection 2.3.

The next proposition is essential in our construction.

Proposition 3.3.1. *The categories $\text{HC}^Q(\mathcal{W}_h), \text{HC}^Q(\mathcal{W}_h^\wedge), \text{HC}^Q(\mathcal{U}_h^\wedge)$ are equivalent. Quasi-inverse equivalences look as follows:*

- $\text{HC}^Q(\mathcal{W}_h) \rightarrow \text{HC}^Q(\mathcal{W}_h^\wedge): \mathcal{N}_h \mapsto \mathcal{N}_h^\wedge$.
- $\text{HC}^Q(\mathcal{W}_h^\wedge) \rightarrow \text{HC}^Q(\mathcal{W}_h): \mathcal{N}'_h \mapsto (\mathcal{N}'_h)_{\mathbb{K}^\times-l.f.}$.
- $\text{HC}^Q(\mathcal{W}_h^\wedge) \rightarrow \text{HC}^Q(\mathcal{U}_h^\wedge): \mathcal{N}'_h \mapsto \mathbf{A}_h^\wedge(\mathcal{N}'_h) := \mathbf{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{N}'_h$.
- $\text{HC}^Q(\mathcal{U}_h^\wedge) \rightarrow \text{HC}^Q(\mathcal{W}_h^\wedge): \mathcal{M}'_h \mapsto (\mathcal{M}'_h)^{\text{ad } V}$.

Proof. To prove that the first two functors are mutually quasiinverse one needs to check that:

- (1) \mathcal{N}_h coincides with the \mathbb{K}^\times -l.f. part of its completion.
- (2) For any \mathcal{N}'_h its \mathbb{K}^\times -l.f. part is dense in \mathcal{N}'_h and is a finitely generated left \mathcal{W}_h -module.

Both claims follow from the fact that \mathcal{W}_h is positively graded. For reader's convenience we give proofs here.

Let us prove (1). For a \mathbb{K}^\times -module N let N^i denote the space of all vectors v with $t.v = t^i v$. Let \mathfrak{m} denote the maximal ideal of 0 in \mathcal{W}_h . Then $\mathfrak{m}^i = \{0\}$ for $i \leq 0$ and $\mathcal{N}_h^i = \{0\}$ for $i \ll 0$. Also $\mathcal{N}_h/\mathfrak{m}^m \mathcal{N}_h$ is a finite dimensional vector space with an algebraic action of \mathbb{K}^\times . For $m > m_0$ we have the exact sequence $(\mathfrak{m}^m \mathcal{N}_h)^i \rightarrow (\mathcal{N}_h/\mathfrak{m}^m \mathcal{N}_h)^i \rightarrow (\mathcal{N}_h/\mathfrak{m}^{m_0} \mathcal{N}_h)^i \rightarrow 0$.

It follows that for any i there is m_0 such that $(\mathcal{N}_h/\mathfrak{m}^m \mathcal{N}_h)^i \xrightarrow{\sim} (\mathcal{N}_h/\mathfrak{m}^{m_0} \mathcal{N}_h)^i$ for $m > m_0$. From this it is easy to deduce (1).

Let us prove (2). First of all, we need to show that the natural projection $(\mathcal{N}'_h)_{\mathbb{K}^\times - l.f.} \rightarrow \mathcal{N}'_h/\mathcal{N}'_h \mathfrak{m}^n$ is surjective for any n . The proof is very similar to the proof of (1). To complete the proof of (2) it is enough to show that $(\mathcal{N}'_h)_{\mathbb{K}^\times - l.f.}$ is finitely generated as a \mathcal{W}_h -module. Since $(\mathcal{N}'_h)_{\mathbb{K}^\times - l.f.}$ is dense in \mathcal{N}'_h , we see that $(\mathcal{N}'_h)_{\mathbb{K}^\times - l.f.}$ generates the left \mathcal{W}_h^\wedge -module \mathcal{N}'_h . Since \mathcal{W}_h^\wedge is a Noetherian algebra, we can choose a finite set of generators. It is easy to see that these elements generate $(\mathcal{N}'_h)_{\mathbb{K}^\times - l.f.}$ (compare with the last paragraph of the proof of Lemma 2.4.2).

Now let us check that the last two functors are quasiinverse equivalences. First, it is clear that $\mathcal{N}'_h = (\mathbf{A}_h^\wedge(\mathcal{N}'_h))^{\text{ad } V}$ for any $\mathcal{N}'_h \in \text{HC}^Q(\mathcal{W}_h^\wedge)$. So we need to verify that the canonical homomorphism $\mathbf{A}_h^\wedge((\mathcal{M}'_h)^{\text{ad } V}) \rightarrow \mathcal{M}'_h$ is an isomorphism for all $\mathcal{M}'_h \in \text{HC}^Q(\mathcal{U}_h^\wedge)$.

Fix a symplectic basis $p_1, q_1, \dots, p_m, q_m$ in V (with $\omega(p_i, p_j) = \omega(q_i, q_j) = 0, \omega(q_i, p_j) = \delta_{ij}$).

Let \mathfrak{n} denote the maximal ideal in \mathbf{A}_h^\wedge (generated by V and \hbar). By Lemma 2.4.4, we have $\mathfrak{n}\mathcal{M}'_h \neq \mathcal{M}'_h$. Choose $m_0 \in \mathcal{M}'_h \setminus \mathfrak{n}\mathcal{M}'_h$. We claim that there is $m \in (\mathcal{M}'_h)^{\text{ad } V}$ such that $m - m_0 \in \mathfrak{n}\mathcal{M}'_h$. At first, we show that there is m'_0 such that $q_1 m'_0 = m'_0 q_1$ and $m'_0 - m_0 \in \mathfrak{n}\mathcal{M}'_h$. There is $m^1 \in \mathcal{M}'_h$ such that $[q_1, m_0] = \hbar^2 m^1$. Then $[q_1, p_1 m^1] = \hbar^2 m_1 + p_1 [q_1, m^1]$. Set $m_1 := m_0 - p_1 m^1$. So $[q_1, m_1] = \hbar^2 p_1 m^2$. Put $m_2 = m_0 - p_1 m^1 - \frac{p_1^2}{2} m^2$, then $[q_1, m_2] = \hbar^2 p_1^2 m^3$ for some $m^3 \in \mathcal{M}'_h$. Define $m^i, i > 3$, in a similar way. Set $m'_0 := m_0 - \sum_{i=1}^\infty \frac{1}{i} p_1^i m^i$. Since \mathcal{M}'_h is complete, m'_0 is well-defined. Also it is clear that $[q_1, m'_0] = 0$. Now apply the same procedure to m'_0 instead of m_0 and p_1 instead of q_1 . We get the element $m''_0 = m'_0 + q_1 m^1 + \frac{1}{2} q_1^2 m^2 + \dots$. By construction, all m^i commute with q_1 . So m''_0 commutes with p_1 and q_1 . Applying this procedure for all q_i and p_i , we get an element m with required properties.

Consider a natural homomorphism $\mathbf{A}_h^\wedge((\mathcal{M}'_h)^{\text{ad } V}) \rightarrow \mathcal{M}'_h$. From the previous paragraph it follows that this homomorphism is surjective. Analogously to the proof of Lemma 3.4.3 in [Lo1], any nonzero \hbar -saturated subbimodule in $\mathbf{A}_h^\wedge((\mathcal{M}'_h)^{\text{ad } V})$ has nonzero intersection with $\text{ad } V$ -invariants, whence the homomorphism is injective. Finally, since $\mathbf{A}_h^\wedge((\mathcal{M}'_h)^{\text{ad } V}) \cong \mathcal{M}'_h$ and \mathcal{M}'_h is a finitely generated (=Noetherian) left $\mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge)$ -module, we see that $(\mathcal{M}'_h)^{\text{ad } V}$ is a Noetherian and hence a finitely generated left \mathcal{W}_h^\wedge -module. So $(\mathcal{M}'_h)^{\text{ad } V} \in \text{HC}^Q(\mathcal{W}_h^\wedge)$. \square

By definition, a functor $\bullet_\dagger : \text{HC}(\mathcal{U}_h) \rightarrow \text{HC}^Q(\mathcal{W}_h)$ is the composition of the completion functor $\text{HC}(\mathcal{U}_h) \rightarrow \text{HC}^Q(\mathcal{U}_h^\wedge)$, see Lemma 2.5.2, and the equivalence $\text{HC}^Q(\mathcal{U}_h^\wedge) \rightarrow \text{HC}^Q(\mathcal{W}_h)$ constructed in Proposition 3.3.1.

The following lemma summarizes the properties of the functor \bullet_\dagger we need.

Lemma 3.3.2. (1) *The functor \bullet_\dagger is exact.*

(2) *The functor \bullet_\dagger is tensor.*

(3) *$\mathcal{M}_\dagger/\hbar\mathcal{M}_\dagger$ is naturally identified with the pull-back of the $\mathbb{K}[\mathfrak{g}^*]$ -module $\mathcal{M}/\hbar\mathcal{M}$ to S .*

(4) *In particular, $(\mathcal{M}_h)_\dagger = 0$ if $V(\mathcal{M}_h) \cap \mathbb{O} = \emptyset$ and $\mathcal{M}_{h\dagger} \in \text{HC}_{fin}^Q(\mathcal{W}_h)$ provided $\mathcal{M}_h \in \text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$. Further, if $\mathcal{M}_h \in \text{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$, then $\text{rk}_{\mathbb{K}[\hbar]} \mathcal{M}_{h\dagger} = \text{mult}_{\overline{\mathbb{O}}} \mathcal{M}_h$.*

Proof. (1) follows from the exactness of the completion functor. (2) follows from the observation that the completion functor as well as the equivalences in Proposition 3.3.1 are tensor.

The construction of \bullet_\dagger implies that we have a \mathbb{K}^\times -equivariant isomorphism $(\mathcal{M}_h/\hbar\mathcal{M}_h)_\chi^\wedge \cong \mathbb{K}[V^*]_0^\wedge \widehat{\otimes} (\mathcal{M}_{h\dagger}/\hbar\mathcal{M}_{h\dagger})_\chi^\wedge$ of $\mathbb{K}[\mathfrak{g}^*]_\chi^\wedge = \mathbb{K}[V^*]_\chi^\wedge \widehat{\otimes} \mathbb{K}[S]_\chi^\wedge$ -modules. This gives rise to a \mathbb{K}^\times -equivariant isomorphism between $(\mathcal{M}_{h\dagger}/\hbar\mathcal{M}_{h\dagger})_\chi^\wedge$ and the pull-back of $(\mathcal{M}_h/\hbar\mathcal{M}_h)_\chi^\wedge$ to S_χ^\wedge .

Then $(\mathcal{M}_h/\hbar\mathcal{M}_h)_\dagger$ (resp., the pull-back of $\mathcal{M}_h/\hbar\mathcal{M}_h$ to S) are just the spaces of \mathbb{K}^\times -l.f. vectors in the modules in the previous sentence, compare with the proof of the first two equivalences in Proposition 3.3.1.

Assertion (4) follows from (3). \square

Now let us produce a functor in the opposite direction. For $\mathcal{M}'_h \in \mathrm{HC}(\mathcal{U}_h^\wedge)$ the subspace $(\mathcal{M}'_h)_{\mathfrak{g}-l.f.} \subset \mathcal{M}'_h$ is \mathbb{K}^\times -stable. Integrate the \mathfrak{g} -action on $(\mathcal{M}'_h)_{\mathfrak{g}-l.f.}$ to a G -action. Define a new \mathbb{K}^\times -action on $(\mathcal{M}'_h)_{\mathfrak{g}-l.f.}$ by composing the existing one with $\gamma(t)^{-1}$. The new action commutes with \mathfrak{g} . Set $(\mathcal{M}'_h)_{l.f.} := [(\mathcal{M}'_h)_{\mathfrak{g}-l.f.}]_{\mathbb{K}^\times-l.f.}$.

Let $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h^\wedge)$ denote the full subcategory in $\mathrm{HC}(\mathcal{U}_h^\wedge)$ consisting of all bimodules \mathcal{M}'_h such that $V(\mathcal{M}'_h)$ is contained in (the completion of) $\overline{\mathbb{O}}$.

Lemma 3.3.3. *If $\mathcal{M}'_h \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h^\wedge)$, then $(\mathcal{M}'_h)_{l.f.} \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$.*

Proof. Set $\mathcal{M}_h := (\mathcal{M}'_h)_{l.f.}$, $M' := \mathcal{M}'_h/\hbar\mathcal{M}'_h$, $M := M'_{\mathfrak{g}-l.f.}$. Let us check that M is a finitely generated $\mathbb{K}[\mathfrak{g}^*]$ -module. Since $\mathcal{M}'_h \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h^\wedge)$, we see that M' is annihilated by some power of the ideal $I(\mathbb{O})$ of $\overline{\mathbb{O}}$. Consider the $I(\mathbb{O})$ -adic filtration on M' . This filtration is finite. Its quotients are objects in $\mathrm{HVB}_{G/(G_\chi)^\circ}^\wedge$. Indeed, the only thing that we need to check that these quotients come equipped with a $(G_\chi)^\circ$ -action. But $(G_\chi)^\circ$ is the semidirect product of its unipotent radical and Q° . The action of the former is uniquely recovered from the action of its Lie algebra.

Let N be one of the quotients. As we checked in Subsection 3.2, assertion (2) of Lemma 3.2.1 and assertion (1) of Proposition 3.2.3, $N_{\mathfrak{g}-l.f.}$ is a finitely generated $\mathbb{K}[G/(G_\chi)^\circ]$ -module and hence a finitely generated $\mathbb{K}[\mathfrak{g}^*]$ -module because $\mathbb{K}[G/(G_\chi)^\circ]$ is finite over \mathfrak{g}^* . Since the functor of taking \mathfrak{g} -l.f. sections (on the category of \mathfrak{g} -modules) is left-exact and all $N_{\mathfrak{g}-l.f.}$ are finitely generated, we see that M is finitely generated.

Consider the exact sequence $0 \rightarrow \mathcal{M}'_h \xrightarrow{\hbar} \mathcal{M}'_h \rightarrow M' \rightarrow 0$. Again, since the functor of taking $(\mathfrak{g}$ - and \mathbb{K}^\times -) l.f. sections is left exact we have the following exact sequence $0 \rightarrow \mathcal{M}_h \xrightarrow{\hbar} \mathcal{M}_h \rightarrow M'_{l.f.}$. It follows that the $\mathbb{K}[\mathfrak{g}^*]$ -module $\mathcal{M}_h/\hbar\mathcal{M}_h$ is finitely generated. Lemma 2.4.4 implies that the \hbar -adic filtration on \mathcal{M}'_h is separated. Hence the same holds for \mathcal{M}_h .

We have the grading $\mathcal{M}_h = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_h^i$. Recall that the multiplication by \hbar increases the degree by 1.

Pick generators m_1, \dots, m_k of the $\mathbb{K}[\mathfrak{g}^*]$ -module $\mathcal{M}_h/\hbar\mathcal{M}_h$ of degrees i_1, \dots, i_k . Lift them to some homogeneous elements $\tilde{m}_1, \dots, \tilde{m}_k \in \mathcal{M}_h$. We claim that these elements generate \mathcal{M}_h .

Let $\underline{\mathcal{M}}_h$ denote the left submodule of \mathcal{M}_h generated by $\tilde{m}_i, i = 1, \dots, k$. It is easy to show by induction that $\mathcal{M}_h = \underline{\mathcal{M}}_h + \hbar^n \mathcal{M}_h$ for any positive integer n . Since the \hbar -adic filtration on \mathcal{M}_h is separated we get that the grading on \mathcal{M}_h is bounded from below. Now the claim in the previous paragraph follows easily.

So $\mathcal{M}_h \in \mathrm{HC}(\mathcal{U}_h)$. It remains to check that $V(\mathcal{M}_h) = \overline{\mathbb{O}}$. Since \mathcal{M}'_h is $\mathbb{K}[[\hbar]]$ -flat, we see that $m \in \mathcal{M}'_h$ is l.f. if and only if so is $\hbar m$. Equivalently, \mathcal{M}_h is an \hbar -saturated subspace in \mathcal{M}'_h . Therefore $\mathcal{M}_h/\hbar\mathcal{M}_h \subset \mathcal{M}'_h/\hbar\mathcal{M}'_h$ and, in particular, $\mathcal{M}_h/\hbar\mathcal{M}_h$ is annihilated by some power of $I(\mathbb{O})$. \square

Similarly to the previous subsection, one can define an action of $C(e) = Q/Q^\circ$ on $(\mathcal{M}'_h)_{l.f.}$ for any $\mathcal{M}'_h \in \mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^\wedge)$. So we get a functor $\mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^\wedge) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$, $\mathcal{M}'_h \mapsto (\mathcal{M}'_h)_{l.f.}^{C(e)}$. Composing this functor with the equivalence $\mathrm{HC}_{fin}^Q(\mathcal{W}_h) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^\wedge)$ from Proposition 2.4.1, we get the functor $\bullet^\dagger : \mathrm{HC}_{fin}^Q(\mathcal{W}_h) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$.

Let us establish a relation between \bullet_{\dagger} and \bullet^{\dagger} .

Proposition 3.3.4. (1) \bullet^{\dagger} is right adjoint to $\bullet_{\dagger} : \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h) \rightarrow \mathrm{HC}_{fin}^Q(\mathcal{W}_h)$. In particular, the functor \bullet^{\dagger} is left exact.
 (2) For any $\mathcal{M} \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$ the kernel and the cokernel of the natural morphism $\mathcal{M}_h \rightarrow (\mathcal{M}_{h\dagger})^{\dagger}$ lie in $\mathrm{HC}_{\partial\mathbb{O}}(\mathcal{U}_h)$.

Proof. (1): Using the equivalence of Proposition 3.3.1 we identify $\mathrm{HC}_{fin}^Q(\mathcal{W}_h)$ with $\mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^{\wedge})$. The functor \bullet_{\dagger} becomes the completion functor \bullet^{\wedge} , while $\bullet^{\dagger} = (\bullet_{l.f.})^{C(e)}$.

Let $\mathcal{M}_h \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}_h)$, $\mathcal{M}'_h \in \mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^{\wedge})$. Pick a morphism $\varphi : \mathcal{M}_h \rightarrow ((\mathcal{M}'_h)_{l.f.})^{C(e)}$. Compose φ with the inclusion $((\mathcal{M}'_h)_{l.f.})^{C(e)} \hookrightarrow \mathcal{M}'_h$. Being a homomorphism of \mathcal{U}_h -modules, the corresponding map $\mathcal{M}_h \rightarrow \mathcal{M}'_h$ is continuous in the $I_{\chi,h}$ -adic topology. So it uniquely extends to a continuous morphism $\psi_{\varphi} : \mathcal{M}_h^{\wedge} \rightarrow \mathcal{M}'_h$.

On the other hand, let $\psi : \mathcal{M}_h^{\wedge} \rightarrow \mathcal{M}'_h$ be a morphism in $\mathrm{HC}_{\overline{\mathbb{O}}}^Q(\mathcal{U}_h^{\wedge})$. We have the natural homomorphism $\mathcal{M}_h \rightarrow \mathcal{M}_h^{\wedge}$. Its image consists of $C(e)$ -invariant $\widetilde{\mathfrak{g}}$ -l.f. vectors. So the image of the composition $\mathcal{M}_h \rightarrow \mathcal{M}_h^{\wedge} \rightarrow \mathcal{M}'_h$ lies in $((\mathcal{M}'_h)_{l.f.})^{C(e)}$. So we get a morphism $\varphi_{\psi} : \mathcal{M}_h \rightarrow ((\mathcal{M}'_h)_{l.f.})^{C(e)}$.

It is easy to see that the assignments $\varphi \mapsto \psi_{\varphi}$ and $\psi \mapsto \varphi_{\psi}$ are inverse to each other.

(2): Let $\mathcal{M}_h^1, \mathcal{M}_h^2$ denote the kernel and the cokernel of the natural morphism $\mathcal{M}_h \rightarrow ((\mathcal{M}_h^{\wedge})_{l.f.})^{C(e)}$. Since the completion functor is exact, we see that $\mathcal{M}_h^{1\wedge} = 0$. By Lemma 3.3.2, $V(\mathcal{M}_h^1) \subset \partial\mathbb{O}$.

Let us prove that $V(\mathcal{M}_h^2) \subset \partial\mathbb{O}$. Since the functor $((\bullet^{\wedge})_{l.f.})^{C(e)}$ (on the category of \mathfrak{g} - and Q -equivariant \mathcal{U}_h -bimodules) is left exact we see that

$$((\mathcal{M}_h^{\wedge})_{l.f.})^{C(e)} / \hbar ((\mathcal{M}_h^{\wedge})_{l.f.})^{C(e)} \hookrightarrow ((\mathcal{M}_h^{\wedge} / \hbar \mathcal{M}_h^{\wedge})_{\mathfrak{g}-l.f.})^{C(e)} = ((\mathcal{M}_h / \hbar \mathcal{M}_h)_{\mathfrak{g}-l.f.}^{\wedge})^{C(e)}.$$

So $\mathcal{M}_h^2 / \hbar \mathcal{M}_h^2$ is embedded into the cokernel of a natural morphism

$$\mathcal{M}_h / \hbar \mathcal{M}_h \rightarrow ((\mathcal{M}_h / \hbar \mathcal{M}_h)_{\mathfrak{g}-l.f.}^{\wedge})^{C(e)}$$

and we need to show that the latter is supported on $\partial\mathbb{O}$.

As in the proof of Lemma 3.3.3, we see that $\mathcal{M}_h / \hbar \mathcal{M}_h$ has a finite filtration by objects of $\mathrm{Coh}^G(\overline{\mathbb{O}})$. So it is enough to show that the cokernel of $M \rightarrow ((M_{\chi}^{\wedge})_{\mathfrak{g}-l.f.})^{C(e)}$ is supported on $\partial\mathbb{O}$ for any $M \in \mathrm{Coh}^G(\overline{\mathbb{O}})$. But, according to assertion (2) of Proposition 3.2.3, $((M_{\chi}^{\wedge})_{\mathfrak{g}-l.f.})^{C(e)} = \Gamma(\mathbb{O}, M|_{\mathbb{O}})$, where $M|_{\mathbb{O}}$ stands for the restriction of M to \mathbb{O} . By Lemma 3.2.1, $\Gamma(\mathbb{O}, M|_{\mathbb{O}})$ is a finitely generated $\mathbb{K}[G/G_{\chi}]$ - (and hence $\mathbb{K}[\overline{\mathbb{O}}]$ -) module. Repeating the argument in the first paragraph of the proof of Proposition 3.2.2, we see that the restriction of $\Gamma(\mathbb{O}, M|_{\mathbb{O}})$ to \mathbb{O} coincides with $M|_{\mathbb{O}}$. So $\Gamma(\mathbb{O}, M|_{\mathbb{O}})/M$ is supported on $\partial\mathbb{O}$. \square

Remark 3.3.5. Of course, we can define \mathcal{N}_h^{\dagger} for arbitrary (not necessarily finite dimensional) bimodule $\mathcal{N}_h \in \mathrm{HC}^Q(\mathcal{W}_h)$ exactly as above. Then \mathcal{N}_h^{\dagger} becomes a \mathfrak{g} -l.f. graded $\mathbb{K}[\hbar]$ -flat \mathcal{U}_h -module. However, we do not prove that \mathcal{N}_h^{\dagger} is finitely generated. The functor \bullet^{\dagger} is still left exact, this can be checked directly. Also we note that the proof of assertion (1) of Proposition 3.3.4 shows that the spaces $\mathrm{Hom}(\mathcal{M}_h, \mathcal{N}_h^{\dagger})$ and $\mathrm{Hom}(\mathcal{M}_{h\dagger}, \mathcal{N}_h)$ (the first Hom-space is taken in the category of graded \mathcal{U}_h -bimodules) are naturally isomorphic. Using this observation and results of Ginzburg, [Gi2], we will see in Subsection 3.5 that \mathcal{N}_h^{\dagger} is finitely generated. We do not use this result in the present paper.

3.4. Construction of functors between $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$, $\mathrm{HC}_{fin}^Q(\mathcal{W})$. Here we construct functors between the categories $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ and $\mathrm{HC}_{fin}^Q(\mathcal{W})$.

Let $\mathcal{M} \in \mathrm{HC}(\mathcal{U})$. Recall the notion of a good filtration on \mathcal{M} introduced in Subsection 2.5. We remark that if $F_i \mathcal{M}, F'_i \mathcal{M}$ are two good filtrations then there are k, l such that $F_{i-k} \mathcal{M} \subset F'_i \mathcal{M} \subset F_{i+l} \mathcal{M}$ for all i . For a good filtration $F_i \mathcal{M}$ on \mathcal{M} set $\mathcal{M}_{\dagger}^F := R_h(\mathcal{M})_{\dagger}/(\hbar-1)R_h(\mathcal{M})_{\dagger}$ (the superscript F indicates the dependence of the bimodule on the choice of F).

Let $\varphi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a homomorphism of two bimodules in $\mathrm{HC}(\mathcal{U})$. Choose good filtrations $F_i^1 \mathcal{M}_1, F_i^2 \mathcal{M}_2$. Replacing $F_i^2 \mathcal{M}_2$ with $F_i'^2 \mathcal{M}_2 := F_{i+k}^2 \mathcal{M}_2$ for sufficiently large k we get $\varphi(F_i^1 \mathcal{M}_1) \subset F_i^2 \mathcal{M}_2$. So φ defines a homomorphism $\varphi_h : R_h(\mathcal{M}_1) \rightarrow R_h(\mathcal{M}_2)$. The homomorphism $\varphi_{h\dagger} : R_h(\mathcal{M}_1)_{\dagger} \rightarrow R_h(\mathcal{M}_2)_{\dagger}$ gives rise to $\varphi_{\dagger}^{F^1, F^2} : \mathcal{M}_{1\dagger}^{F^1} \rightarrow \mathcal{M}_{2\dagger}^{F^2}$. Clearly, if $\varphi : \mathcal{M}_1 \rightarrow \mathcal{M}_2, \psi : \mathcal{M}_2 \rightarrow \mathcal{M}_3$ are two homomorphisms and F^i is a good filtration on $\mathcal{M}_i, i = 1, 2, 3$, with $\varphi(F_i^1 \mathcal{M}_1) \subset F_i^2 \mathcal{M}_2, \psi(F_i^2 \mathcal{M}_2) \subset F_i^3 \mathcal{M}_3$, then

$$(3.1) \quad (\psi \circ \varphi)_{\dagger}^{F^1, F^3} = \psi_{\dagger}^{F^2, F^3} \circ \varphi_{\dagger}^{F^1, F^2}.$$

For two good filtrations F, F' on \mathcal{M} with $F_i \mathcal{M} \subset F'_i \mathcal{M}$ consider the morphism $\mathrm{id}_{\dagger}^{F, F'} : \mathcal{M}_{\dagger}^F \rightarrow \mathcal{M}_{\dagger}^{F'}$. We claim that this is an isomorphism. Indeed, for sufficiently large k we have an inclusion $F'_i \mathcal{M} \subset F_{i+k} \mathcal{M}$. Since $\mathrm{id}_{\dagger}^{F, F'+k}$ is the identity morphism of $\mathcal{M}_{\dagger}^F = \mathcal{M}_{\dagger}^{F'+k}$, (3.1) implies that $\mathrm{id}_{\dagger}^{F, F'}$ has a left inverse. Similarly, it also has a right inverse. Using the isomorphisms $\mathrm{id}_{\dagger}^{F, F'}$ and their inverses we can identify all \mathcal{M}_{\dagger}^F (the identification does not depend on the choice of intermediate filtrations thanks to (3.1)). Similarly, we see that $\varphi_{\dagger}^{F^1, F^2}$ is also independent on the choice of F^1, F^2 modulo the identifications we made.

Summarizing, we get a functor $\bullet_{\dagger} : \mathrm{HC}(\mathcal{U}) \rightarrow \mathrm{HC}^Q(\mathcal{W})$.

Now let us construct a functor $\bullet^{\dagger} : \mathrm{HC}_{fin}^Q(\mathcal{W}) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$. For a module $\mathcal{N} \in \mathrm{HC}_{fin}^Q(\mathcal{W})$ define a filtration $F_i \mathcal{N}$ by setting $F_{-1} \mathcal{N} = \{0\}, F_0 \mathcal{N} = \mathcal{N}$. Since $K_0 \mathcal{W} = K_1 \mathcal{W} = \mathbb{K} (\mathbb{K}[S]$ has no component of degree 1), we get $[K_i \mathcal{W}, F_j \mathcal{N}] \subset F_{i+j-2} \mathcal{N}$.

Put $\mathcal{N}^{\dagger} := R_h(\mathcal{N})^{\dagger}/(\hbar-1)$. Then \mathcal{N}^{\dagger} comes equipped with a good filtration. Every homomorphism $\varphi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ gives rise to $\varphi^{\dagger} : \mathcal{N}_1^{\dagger} \rightarrow \mathcal{N}_2^{\dagger}$. The data $\mathcal{N} \mapsto \mathcal{N}^{\dagger}, \varphi \mapsto \varphi^{\dagger}$ constitute a functor.

Interpreting results of the previous subsection in the present situation, we get the following proposition.

- Proposition 3.4.1.** (1) *The functor $\bullet_{\dagger} : \mathrm{HC}(\mathcal{U}) \rightarrow \mathrm{HC}^Q(\mathcal{W})$ is exact and maps $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ to $\mathrm{HC}_{fin}^Q(\mathcal{W})$ and $\mathrm{HC}_{\partial \mathbb{O}}(\mathcal{U})$ to zero.*
(2) *The functor $\bullet_{\dagger} : \mathrm{HC}(\mathcal{U}) \rightarrow \mathrm{HC}^Q(\mathcal{W})$ is tensor.*
(3) *$\dim \mathcal{M}_{\dagger} = \mathrm{mult}_{\overline{\mathbb{O}}} \mathcal{M}$ for any $\mathcal{M} \in \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$.*
(4) *The functor \bullet^{\dagger} is right adjoint to the restriction of \bullet_{\dagger} to $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$.*
(5) *The kernel and the cokernel of the natural morphism $\mathcal{M} \rightarrow (\mathcal{M}_{\dagger})^{\dagger}$ lie in $\mathrm{HC}_{\partial \mathbb{O}}(\mathcal{U})$.*

Proof. Assertions (1) and (3) follow directly from Lemma 3.3.2. Also using Lemma 3.3.2 one can reduce assertion (2) to the following claim:

Let \mathcal{A} be a filtered algebra and M, N be filtered \mathcal{A} -bimodules. Then the natural homomorphism $R_h(M) \otimes_{R_h(\mathcal{A})} R_h(N)/(\hbar-1)R_h(M) \otimes_{R_h(\mathcal{A})} R_h(N) \rightarrow M \otimes_{\mathcal{A}} N$ is an isomorphism.

The claim follows from the observation that $R_h(M) \otimes_{R_h(\mathcal{A})} R_h(N)/(\hbar-1)R_h(M) \otimes_{R_h(\mathcal{A})} R_h(N)$ is naturally identified with $\mathcal{A} \otimes_{R_h(\mathcal{A})} R_h(M) \otimes_{R_h(\mathcal{A})} R_h(N) \otimes_{R_h(\mathcal{A})} \mathcal{A}$. But the latter is nothing else but $M \otimes_{R_h(\mathcal{A})} N = M \otimes_{\mathcal{A}} N$.

Let us derive assertion (4) from the corresponding assertion of Proposition 3.3.4. Pick $\varphi \in \text{Hom}(\mathcal{M}, \mathcal{N}^\dagger)$, where $\mathcal{M} \in \text{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$, $\mathcal{N} \in \text{HC}_{fin}^Q(\mathcal{W})$. There are good filtrations $F_i \mathcal{M}$, $F'_i \mathcal{N}$ on \mathcal{M}, \mathcal{N} respectively such that $\varphi(F_i \mathcal{M}) \subset F'_i \mathcal{N}^\dagger$ (here in the right hand side $F'_i \mathcal{N}^\dagger$ is a good filtration arising from $F'_i \mathcal{N}$, see the construction above). So φ gives rise to a morphism $\varphi_h : R_h(\mathcal{M}) \rightarrow R_h(\mathcal{N}^\dagger) = (R_h(\mathcal{N}))^\dagger$ in $\text{HC}(\mathcal{U}_h)$. The corresponding homomorphism $\psi : R_h(\mathcal{M})_\dagger \rightarrow R_h(\mathcal{N})$ gives rise to $\psi : \mathcal{M}_\dagger \rightarrow \mathcal{N}$. Similarly to the construction of the functors above, the morphism ψ does not depend on the choice of good filtrations. So we get a natural map $\text{Hom}(\mathcal{M}, \mathcal{N}^\dagger) \rightarrow \text{Hom}(\mathcal{M}_\dagger, \mathcal{N})$. Similarly, we get the inverse map.

Assertion (5) follows now directly from assertion (2) of Proposition 3.3.4. \square

Remark 3.4.2. It follows easily from the construction that for $\mathcal{J} \in \mathfrak{Id}_{\overline{\mathbb{O}}}(\mathcal{U})$ the definition of \mathcal{J}_\dagger given here is the same as in Subsection 3.1.

Remark 3.4.3. One can define \mathcal{N}^\dagger for an arbitrary object $\mathcal{N} \in \text{HC}^Q(\mathcal{W})$ using the same procedure as above and Remark 3.3.5. We still have the natural isomorphism $\text{Hom}(\mathcal{M}, \mathcal{N}^\dagger) \cong \text{Hom}(\mathcal{M}_\dagger, \mathcal{N})$ and hence the natural morphism $\mathcal{M} \rightarrow (\mathcal{M}_\dagger)^\dagger$. The functor \bullet^\dagger is left exact.

In particular, for an ideal $\mathcal{I} \subset \mathcal{W}$ we have defined the ideal $\mathcal{I}^\dagger \subset \mathcal{U}$, see Subsection 3.1. One can show that the functor \bullet^\dagger maps \mathcal{W} to \mathcal{U} . And so for a $C(e)$ -stable ideal \mathcal{I} both definitions of \mathcal{I}^\dagger agree. We are not going to use this. Instead we observe the following. By the definition of $\mathcal{I}^\dagger \subset \mathcal{U}$ given in Subsection 3.1, it is nothing else but the preimage of $\mathcal{I}^\dagger \subset \mathcal{W}^\dagger$ under the natural homomorphism $\mathcal{U} \rightarrow \mathcal{W}^\dagger = (\mathcal{U}_\dagger)^\dagger$ provided \mathcal{I} is Q -stable. In particular, it follows that $\mathcal{I}^\dagger \subset \mathcal{U}$ is the kernel of the natural map $\mathcal{U} \rightarrow \mathcal{W}^\dagger / \mathcal{I}^\dagger \hookrightarrow (\mathcal{W} / \mathcal{I})^\dagger$.

Remark 3.4.4. We can use the previous remark and results of Borho and Kraft, [BoKr] to prove the Joseph irreducibility theorem, [Jo], see the discussion at the end of Subsection 3.1. Other proofs can be found in [V], [Gi1].

If \mathcal{I} is a $C(e)$ -stable ideal of finite codimension in \mathcal{W} , then, thanks, to the previous remark $V(\mathcal{U} / \mathcal{I}^\dagger) \subset V((\mathcal{W} / \mathcal{I})^\dagger) \subset \overline{\mathbb{O}}$. On the other hand, $(\mathcal{I}^\dagger)_\dagger \subset \mathcal{I}$ by assertion (ii) of Theorem 3.1.1. So $V(\mathcal{U} / \mathcal{I}^\dagger) = \overline{\mathbb{O}}$.

Now we are ready to rederive the Joseph irreducibility theorem. Suppose \mathcal{J} is primitive ideal in \mathcal{U} such that $\overline{\mathbb{O}}$ is an irreducible component of $V(\mathcal{U} / \mathcal{J})$ of maximal dimension. Therefore \mathcal{J}_\dagger has finite codimension in \mathcal{W} . By assertion (ii) of Theorem 3.1.1, $\mathcal{J} \subset (\mathcal{J}_\dagger)^\dagger$. Now Corollar 3.6, [BoKr], implies that $\mathcal{J} = (\mathcal{J}_\dagger)^\dagger$. So $V(\mathcal{U} / \mathcal{J}) = \overline{\mathbb{O}}$.

To finish the subsection we will prove a generalization of [Lo1], Proposition 3.4.6, which was conjectured by McGovern in [McG].

Proposition 3.4.5. *Let \mathcal{A} be a Dixmier algebra (i.e. an algebra over \mathcal{U} that is an HC bimodule with respect to the left and right multiplications by elements of \mathcal{U}) such that $V(\mathcal{A}) = \overline{\mathbb{O}}$. Suppose, in addition, that \mathcal{A} is prime. Then $\text{Grk}(\mathcal{A}) \leq \sqrt{\text{mult}_{\overline{\mathbb{O}}}(\mathcal{A})}$.*

Proof. It is easy to see that \mathcal{A} admits a good filtration that is also an algebra filtration. From the construction we see that $\mathcal{A}_\dagger, (\mathcal{A}_\dagger)^\dagger$ have natural algebra structures and the natural morphism $\mathcal{A} \rightarrow (\mathcal{A}_\dagger)^\dagger$ is a homomorphism of algebras. Analogously to the proof of Proposition 3.4.6 in [Lo1], we have a homomorphism $\psi : \mathcal{A} \rightarrow \mathcal{B} \otimes \mathcal{A}_\dagger$, where \mathcal{B} is a certain completely prime (=without zero divisors) algebra. More precisely, $\mathcal{B} := (\mathbf{A}_h^\wedge)_{\mathbb{K}^\times - l.f. / (\hbar - 1)}$ and ψ is obtained from a natural homomorphism $\mathcal{A}_h \rightarrow \mathcal{A}_h^\wedge = \mathbf{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{A}_{h\dagger}$ (where $\mathcal{A}_h = R_h(\mathcal{A})$ for an appropriate grading on \mathcal{A}) by taking quotient of $\mathbb{K}^\times - l.f.$ -finite parts by $\hbar - 1$. In particular, for any ideal $\mathcal{I} \subset \mathcal{A}_\dagger$ we have $\mathcal{I}^\dagger = \psi^{-1}(\mathcal{B} \otimes \mathcal{I})$.

Let \mathcal{I} be a minimal prime ideal of 0 in \mathcal{A}_\dagger . Set $\mathcal{J} := \psi^{-1}(\mathcal{B} \otimes \mathcal{I})$ in \mathcal{A} . In other words, \mathcal{J} is the preimage of $\mathcal{I}^\dagger \subset (\mathcal{A}_\dagger)^\dagger$ under $\mathcal{A} \rightarrow (\mathcal{A}_\dagger)^\dagger$. We are going to show that $\mathcal{J} = \{0\}$.

Assume the converse. Since the algebra \mathcal{A} is prime, we can apply results of Borho and Kraft, [BoKr] to see that \mathcal{A}/\mathcal{J} is supported on $\partial\mathbb{O}$, equivalently, $\mathcal{A}_\dagger = \mathcal{J}_\dagger$. However, by assertion (ii) of Theorem 3.1.1, $\mathcal{J}_\dagger \subset \mathcal{I}$, contradiction. So we have an embedding $\mathcal{A} \hookrightarrow \mathcal{B} \otimes (\mathcal{A}_\dagger/\mathcal{I})$.

Now, similarly to [Lo1], $\text{Grk}(\mathcal{A}) \leq \text{Grk}(\mathcal{B} \otimes (\mathcal{A}_\dagger/\mathcal{I})) = \text{Grk}(\mathcal{A}_\dagger/\mathcal{I}) \leq \sqrt{\dim \mathcal{A}_\dagger} = \sqrt{\text{mult}_{\mathbb{O}}(\mathcal{A})}$. \square

3.5. Comparison with Ginzburg's construction. Ginzburg, [Gi2], defined a functor $\text{HC}_{\mathbb{O}}(\mathcal{U}) \rightarrow \text{HC}_{fin}^Q(\mathcal{W})$ in the following way: $\mathcal{M} \mapsto (\mathcal{M}/\mathcal{M}\mathfrak{m}_\chi)^{\text{adm}}$ (to see the action of Q one needs to prove that the natural homomorphism $(\mathcal{M}/\mathfrak{g}_{\leq -2, \chi})^{\text{ad } \mathfrak{g}_{\leq -1}} \rightarrow (\mathcal{M}/\mathfrak{m}_\chi \mathcal{M})^{\text{adm}}$ is an isomorphism, this can be done similarly to [GG], Subsection 5.5). Below in this subsection we will check that Ginzburg's functor coincides with ours. In particular, on the language of the quantum Hamiltonian reduction one has $\mathcal{J}_\dagger = (\mathcal{J}/\mathcal{J}\mathfrak{m}_\chi)^{\text{adm}}$.

Recall the algebras $\mathcal{U}^\heartsuit := (\mathcal{U}_h^\wedge)_{\mathbb{K}^\times - \text{l.f.}}/(\hbar - 1)$, $\mathbf{A}(\mathcal{W})^\heartsuit := (\mathbf{A}_h^\wedge(\mathcal{W}_h^\wedge))_{\mathbb{K}^\times - \text{l.f.}}/(\hbar - 1)$ introduced in [Lo1]. Let $\Phi : \mathcal{U}^\heartsuit \rightarrow \mathbf{A}(\mathcal{W})^\heartsuit$ be the isomorphism induced by Φ_h . Now let \mathcal{M} be a HC \mathcal{U} -bimodule. Choosing a good filtration on \mathcal{M} , we get a HC \mathcal{U}_h -bimodule $\mathcal{M}_h = R_h(\mathcal{M})$. Set $\mathcal{M}^\heartsuit := (\mathcal{M}_h^\wedge)_{\mathbb{K}^\times - \text{l.f.}}/(\hbar - 1)$. Analogously to the previous subsection, the \mathcal{U}^\heartsuit -bimodule \mathcal{M}^\heartsuit does not depend on the choice of a filtration on \mathcal{M} . Moreover, from the construction of \mathcal{M}_\dagger it follows that $\mathcal{M}^\heartsuit = \mathbf{A}(\mathcal{M}_\dagger)^\heartsuit$. So the \mathcal{W} -bimodule \mathcal{M}_\dagger is nothing else but $(\mathcal{M}^\heartsuit)^{\text{ad } V} \cong (\mathcal{M}^\heartsuit/\mathcal{M}^\heartsuit \mathfrak{m})^{\text{adm}}$, where we consider \mathfrak{m} as a lagrangian subspace in V .

The embedding $\mathcal{U} \hookrightarrow \mathcal{U}^\heartsuit$ gives rise to a map $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi \rightarrow \mathcal{U}^\heartsuit/\mathcal{U}^\heartsuit \mathfrak{m}_\chi$. As we have seen in [Lo1], the paragraph preceding Remark 3.2.7, $\mathcal{U}^\heartsuit = \mathcal{U} + \mathcal{U}^\heartsuit \mathfrak{m}_\chi$. Also it is clear from the construction there that $\mathcal{U} \cap \mathcal{U}^\heartsuit \mathfrak{m}_\chi = \mathcal{U}\mathfrak{m}_\chi$ (this was used implicitly in the proof of Corollary 3.3.3 in [Lo1]). So the natural homomorphism $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{adm}_\chi} \rightarrow (\mathcal{U}^\heartsuit/\mathcal{U}^\heartsuit \mathfrak{m}_\chi)^{\text{adm}_\chi} = \mathcal{W}$ is an isomorphism of filtered algebras.

So we have a functorial homomorphism $\iota : (\mathcal{M}/\mathcal{M}\mathfrak{m}_\chi)^{\text{adm}_\chi} \rightarrow (\mathcal{M}^\heartsuit/\mathcal{M}^\heartsuit \mathfrak{m}_\chi)^{\text{adm}_\chi} = \mathcal{M}_\dagger$ of \mathcal{W} -modules. This homomorphism preserves natural (Kazhdan) filtrations on the bimodules. Let us check that ι is an isomorphism. By assertion (3) of Lemma 3.3.2 and part (i) of Theorem 4.1.4 in [Gi2], we have $\text{gr}(\mathcal{M}/\mathcal{M}\mathfrak{m}_\chi)^{\text{adm}} \cong \text{gr } \mathcal{M}|_S \cong \text{gr } \mathcal{M}_\dagger$. Moreover, the corresponding isomorphism $\text{gr}(\mathcal{M}/\mathcal{M}\chi)^{\text{adm}} \rightarrow \text{gr } \mathcal{M}_\dagger$ coincides with $\text{gr } \iota$. Since the gradings on both modules are bounded from below, we see that ι is an isomorphism.

Remark 3.5.1. As Ginzburg proved in [Gi2], Theorem 4.2.2, there is a right adjoint functor to $\bullet_\dagger : \text{HC}(\mathcal{U}) \rightarrow \text{HC}(\mathcal{W})$ (he did not considered Q -equivariant structures). We can consider the functor \bullet^\heartsuit from $\text{HC}(\mathcal{W})$ to the category of \mathfrak{g} -l.f. \mathcal{U} -bimodules corresponding to taking l.f. sections on the homogeneous level (without taking $C(e)$ -invariants). Similarly to Remark 3.4.3, we see that $\text{Hom}(\mathcal{M}, \mathcal{N}^\heartsuit) = \text{Hom}(\mathcal{M}_\dagger, \mathcal{N})$. Thanks to Ginzburg's result, the image of \bullet^\heartsuit lies in $\text{HC}(\mathcal{U})$. In particular, we see that the image of \bullet^\dagger lies in $\text{HC}(\mathcal{U})$. We are not going to use this result below.

4. PROOF OF THEOREMS 1.2.2, 1.3.1

4.1. Surjectivity theorem. The following theorem completes the proof of Theorem 1.2.2 and essentially implies the most non-trivial part of Theorem 1.3.1, assertion 5.

Theorem 4.1.1. *Let $\mathcal{M} \in \mathrm{HC}(\mathcal{U})$ and let $\mathcal{N} \subset \mathcal{M}_\dagger$ be a Q -stable subbimodule of finite codimension. Let \mathcal{N}^\ddagger stand for the preimage of $\mathcal{N}^\dagger \subset (\mathcal{M}_\dagger)^\dagger$ in \mathcal{M} (see Remark 3.4.3). Then $(\mathcal{N}^\ddagger)_\dagger = \mathcal{N}$.*

Proof. Examining the construction of the functors we see that the claim of the theorem follows from the following claim:

- (*) Let $\mathcal{M}_\hbar \in \mathrm{HC}(\mathcal{U}_\hbar)$ and \mathcal{N}'_\hbar be a \mathfrak{g} - (with respect to the action $\xi \mapsto \frac{1}{\hbar^2}[\xi, \cdot]$), \mathbb{K}^\times - and Q -stable (but not necessary \hbar -saturated) \mathcal{U}_\hbar^\wedge -subbimodule in \mathcal{M}_\hbar^\wedge . Then \mathcal{N}'_\hbar is the closure of its preimage \mathcal{N}_\hbar in \mathcal{M}_\hbar .

For a \mathcal{U}_\hbar^\wedge -subbimodule $\mathcal{N}'_\hbar \subset \mathcal{M}_\hbar^\wedge$ we define its \hbar -saturation $\widehat{\mathcal{N}}'_\hbar$ as the subset of \mathcal{M}_\hbar^\wedge consisting of all elements $m \in \mathcal{M}_\hbar^\wedge$ with $\hbar^k m \in \mathcal{N}'_\hbar$. Since \mathcal{M}_\hbar^\wedge is Noetherian there is $N \in \mathbb{N}$ with $\hbar^N \widehat{\mathcal{N}}'_\hbar \subset \mathcal{N}'_\hbar$.

Let us show that if (*) holds for $\widehat{\mathcal{N}}'_\hbar$, then it holds for \mathcal{N}'_\hbar . So suppose that the completion $\widehat{\mathcal{N}}'_\hbar$ of the preimage $\widehat{\mathcal{N}}_\hbar$ of $\widehat{\mathcal{N}}'_\hbar$ coincides with $\widehat{\mathcal{N}}'_\hbar$.

Consider the subbimodule $\mathcal{N}'_\hbar/\mathcal{N}_\hbar^\wedge \subset (\widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar)^\wedge$. We remark that $\widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar$ is embedded into $\widehat{\mathcal{N}}'_\hbar/\mathcal{N}'_\hbar$ and hence is annihilated by some power of \hbar and is supported on $\overline{0}$. Let $I_\hbar(\mathbb{O})$ denote the preimage of $I(\mathbb{O})$ in \mathcal{U}_\hbar . We see that some power of $I_\hbar(\mathbb{O})$ annihilates $\widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar$. Let M denote the (no matter, left or right) annihilator of $I_\hbar(\mathbb{O})$ in $\widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar$. By Corollary 2.4.3, the annihilator of $I_\hbar(\mathbb{O})$ in $(\widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar)^\wedge$ coincides with M^\wedge . In particular, $N' := \mathcal{N}'_\hbar \cap M \neq 0$. Both M^\wedge and N' are objects in $\mathrm{HVB}_{G/G_\chi}^\wedge$. It follows from Propositions 3.2.2, 3.2.3 that N' coincides with the completion of its preimage N in M . Consider the preimage $\widetilde{\mathcal{N}}_\hbar$ of N under the projection $\widehat{\mathcal{N}}_\hbar \rightarrow \widehat{\mathcal{N}}_\hbar/\mathcal{N}_\hbar$. Then $\widetilde{\mathcal{N}}_\hbar^\wedge \subset \mathcal{N}'_\hbar$. Therefore $\widetilde{\mathcal{N}}_\hbar = \mathcal{N}_\hbar$. Contradiction.

In particular, we see that (*) holds for \mathcal{N}'_\hbar provided $\hbar^k \mathcal{M}_\hbar^\wedge \subset \mathcal{N}'_\hbar$. Also we see that it is enough to prove (*) for \hbar -saturated \mathcal{N}'_\hbar . Below \mathcal{N}'_\hbar is assumed to be \hbar -saturated.

Set $\mathcal{N}'_{\hbar,k} = \mathcal{N}'_\hbar + \hbar^{k+1} \mathcal{M}_\hbar^\wedge$. Let $\mathcal{N}_{\hbar,k}$ denote the preimage of $\mathcal{N}'_{\hbar,k}$ in \mathcal{M}_\hbar . By the above, $\mathcal{N}_{\hbar,k}^\wedge = \mathcal{N}'_{\hbar,k}$. On the other hand, $\bigcap_k \mathcal{N}'_{\hbar,k} = \mathcal{N}'_\hbar$ because \mathcal{N}'_\hbar is closed in \mathcal{M}_\hbar^\wedge , see Lemma 2.4.4. Therefore $\bigcap_k \mathcal{N}_{\hbar,k} = \mathcal{N}_\hbar$. Since $\mathcal{N}'_\hbar \subset \mathcal{M}_\hbar^\wedge$ is \hbar -saturated, we see that so is $\mathcal{N}_\hbar \subset \mathcal{M}_\hbar$. Replace \mathcal{M}_\hbar with $\mathcal{M}_\hbar/\mathcal{N}_\hbar$ and \mathcal{N}'_\hbar with $\mathcal{N}'_\hbar/\mathcal{N}_\hbar^\wedge$. So we may and will assume that $\mathcal{N}_\hbar = \{0\}$. We need to check that $\mathcal{N}'_\hbar = \{0\}$. Assume the converse.

Set $T_k := \mathcal{N}_{\hbar,k}/\mathcal{N}_{\hbar,k+1}$. By definition, this is a $\mathcal{U}_\hbar/(\hbar^{k+2})$ -module. However, it is easy to see that

$$(4.1) \quad \hbar \mathcal{N}_{\hbar,k} \subset \mathcal{N}_{\hbar,k+1},$$

So \hbar acts trivially on T_k and T_k is a $\mathbb{K}[\mathfrak{g}^*]$ -module. Moreover, (4.1) implies that the multiplication by \hbar induces a homomorphism $T_k \rightarrow T_{k+1}$ of $\mathbb{K}[\mathfrak{g}^*]$ -modules also denoted by \hbar . So $T := \bigoplus_{i=0}^\infty T_i$ becomes a $\mathbb{K}[\mathfrak{g}^*][\hbar]$ -module.

Set $C := [\mathcal{M}_\hbar^\wedge/\mathcal{N}'_{\hbar,0}]_{l.f.}$. The argument of Lemma 3.3.3 implies that C is a finitely generated $\mathbb{K}[\mathfrak{g}^*]$ -module.

Lemma 4.1.2. *There is an embedding $T \hookrightarrow C[\hbar]$ of $\mathbb{K}[\mathfrak{g}^*][\hbar]$ -modules.*

Proof. We will construct embeddings $\iota_i : T_i \hookrightarrow C, i = 0, 1, \dots$, such that $\iota_{i+1}(\hbar x) = \iota_i(x)$ for all $x \in T_i$.

Since $\mathcal{M}_\hbar^\wedge/\mathcal{N}'_\hbar$ is $\mathbb{K}[\hbar]$ -flat, have the following exact sequence

$$(4.2) \quad 0 \rightarrow \mathcal{M}_\hbar^\wedge/\mathcal{N}'_{\hbar,0} \rightarrow \mathcal{M}_\hbar^\wedge/\mathcal{N}'_{\hbar,k+1} \rightarrow \mathcal{M}_\hbar^\wedge/\mathcal{N}'_{\hbar,k} \rightarrow 0,$$

where the first map is the multiplication by \hbar^{k+1} , and thus an exact sequence

$$(4.3) \quad 0 \rightarrow C \rightarrow (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k+1})_{l.f.} \rightarrow (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k})_{l.f.}$$

There is a natural inclusion

$$T_k \hookrightarrow \mathcal{M}_{\hbar}/\mathcal{N}_{\hbar,k+1} \hookrightarrow (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k+1})_{l.f.},$$

whose image in $(\mathcal{M}_{\hbar,k}^{\wedge}/\mathcal{N}'_{\hbar,k})_{l.f.}$ is zero. So we get a $\mathbb{K}[\mathfrak{g}^*]$ -module embedding $T_k \hookrightarrow C$.

The claim that these embeddings are compatible with the multiplication by \hbar stems from the following commutative diagram.

$$\begin{array}{ccccc}
 0 & & & & 0 \\
 \downarrow & & & & \downarrow \\
 C & \xrightarrow{\text{id}} & C & & C \\
 \swarrow & \xrightarrow{\hbar} & \searrow & & \swarrow \\
 & T_k & & T_{k+1} & \\
 \swarrow & \xrightarrow{\hbar} & \searrow & & \swarrow \\
 (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k+1})_{l.f.} & \xrightarrow{\hbar} & (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k+2})_{l.f.} & & \\
 \downarrow & & \downarrow & & \\
 (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k})_{l.f.} & \xrightarrow{\hbar} & (\mathcal{M}_{\hbar}^{\wedge}/\mathcal{N}'_{\hbar,k+1})_{l.f.} & &
 \end{array}$$

□

Let us complete the proof of Theorem 4.1.1. Being a submodule in a Noetherian module, the $\mathbb{K}[\mathfrak{g}^*][\hbar]$ -module T is finitely generated. It follows that there is $m > 0$ such that $T_i = \hbar^{i-k}T_k$ for all $i > k$. This implies

$$(4.4) \quad \mathcal{N}_{\hbar,i} = \hbar^{i-k}\mathcal{N}_{\hbar,k} + \mathcal{N}_{\hbar,i+1}.$$

Now recall that \mathcal{M}_{\hbar} is graded, the grading is bounded from below, and all graded components are finite dimensional. All $\mathcal{N}_{\hbar,i}$ are graded sub-bimodules in \mathcal{M}_{\hbar} . (4.4) implies that for any k the k -th graded component of \mathcal{N}_{\hbar} coincides with that of $\mathcal{N}_{\hbar,i}$ for sufficiently large i . Also (4.4) implies that the inverse sequence of the projections of $\mathcal{N}_{\hbar,i}$ to $\mathcal{M}_{\hbar,0}$ stabilizes. So we can find k such that the k -th graded component of $\mathcal{N}_{\hbar,i}$ is nonzero for all i . It follows that $\mathcal{N}_{\hbar} \neq \{0\}$. Contradiction. □

4.2. Completing the proofs. Below for $\mathcal{I} \subset \mathcal{W}$ the notation \mathcal{I}^{\dagger} means an ideal in \mathcal{U} (so that we follow the conventions of [Lo1]).

Theorem 1.2.2 follows directly from Theorem 4.1.1 with $\mathcal{M} = \mathcal{U}$.

Proof of Conjecture 1.2.1. Thanks to [Lo1], Theorem 1.2.2(viii), we need to prove that Q acts transitively on the set of minimal prime ideals $\mathcal{I}_1, \dots, \mathcal{I}_l$ of \mathcal{J}_{\dagger} , where $\mathcal{J} \in \mathfrak{Id}_{\overline{0}}(\mathcal{U})$ is primitive. The ideal $\cap_{\gamma \in C(e)} \gamma \mathcal{I}_1$ is Q -stable and so, by Theorem 1.2.2, $\mathcal{J}_{\dagger}^1 = \cap_{\gamma \in C(e)} \gamma \mathcal{I}_1$, where $\mathcal{J}^1 := (\cap_{\gamma \in C(e)} \gamma \mathcal{I}_1)^{\dagger}$. But $\mathcal{J} = \mathcal{I}_1^{\dagger} \supset \mathcal{J}^1 \supset \mathcal{J}$. We deduce that $\cap_{\gamma \in C(e)} \gamma \mathcal{I}_1 = \cap_{i=1}^l \mathcal{I}_i$. From this it is easy to see that any \mathcal{I}_i has the form $\gamma \mathcal{I}_1$. □

Proof of Theorem 1.3.1. Assertions (1),(2),(3) follow from Proposition 3.3.4.

Let us check assertion (4) for the left annihilators (right ones are completely analogous). Set $\mathcal{J} := \text{LAnn}_{\mathcal{U}}(\mathcal{M}), \mathcal{I} := \text{LAnn}_{\mathcal{W}}(\mathcal{M}_{\dagger})$. Since \bullet_{\dagger} is an exact tensor functor, we see that

$\mathcal{J}_\dagger \subset \mathcal{I}$. On the other hand, by Theorem 1.2.2, $\mathcal{I} := \tilde{\mathcal{J}}_\dagger$ for $\tilde{\mathcal{J}} = \mathcal{I}^\dagger$. Again, since \bullet_\dagger is a tensor functor, we see that $(\tilde{\mathcal{J}}\mathcal{M})_\dagger = \mathcal{I}\mathcal{M}_\dagger = 0$. So $\tilde{\mathcal{J}}\mathcal{M} \in \mathrm{HC}_{\partial\mathbb{O}}(\mathcal{U})$. Set $\mathcal{J}^1 := \mathrm{LAnn}_{\mathcal{U}}(\tilde{\mathcal{J}}\mathcal{M})$. We have $\mathcal{J}^1\tilde{\mathcal{J}} \subset \mathcal{J}$. So $\mathcal{J}_\dagger^1\mathcal{I} = (\mathcal{J}^1\tilde{\mathcal{J}})_\dagger \subset \mathcal{J}_\dagger$. But $\mathcal{J}_\dagger^1 = \mathcal{W}$ so $\mathcal{I} \subset \mathcal{J}_\dagger$.

Proceed to the proof of (5). By assertion (1) of Proposition 3.3.4, \bullet_\dagger descends to $\mathrm{HC}_{\mathbb{O}}(\mathcal{U})$. Abusing the notation we write \bullet^\dagger for the composition of $\bullet^\dagger : \mathrm{HC}_{fin}^Q(\mathcal{W}) \rightarrow \mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U})$ and the projection $\mathrm{HC}_{\overline{\mathbb{O}}}(\mathcal{U}) \rightarrow \mathrm{HC}_{\mathbb{O}}(\mathcal{U})$. This functor is right adjoint to $\bullet_\dagger : \mathrm{HC}_{\mathbb{O}}(\mathcal{U}) \rightarrow \mathrm{HC}_{fin}^Q(\mathcal{W})$. By assertion (5) of Proposition 3.3.4, \bullet^\dagger is left inverse to \bullet_\dagger . From here using some abstract nonsense we see that \bullet_\dagger is an equivalence onto its image.

The claim that the image of \bullet_\dagger is closed under taking quotients follows from Theorem 4.1.1. Then the image is automatically closed with respect to taking subquotients. \square

Proof of Corollary 1.3.2. Thanks to assertion 5 of Theorem 1.3.1, it is enough to show that \mathcal{M}_\dagger is completely reducible in $\mathrm{HC}_{fin}^Q(\mathcal{W})$. By assertion 4 and Theorem 1.2.2 (together with the proof of Conjecture 1.2.1) the left and right annihilators of \mathcal{M}_\dagger are intersections of primitive ideals of finite codimension. So \mathcal{W} acts \mathcal{M}_\dagger via an epimorphism to the direct sums of matrix algebras. Therefore \mathcal{M}_\dagger is completely reducible. \square

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